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A PULSE METHOD OF FORMULATING A MATHEMATICAL
MODEL OF A PHYSICAL SYSTEM

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A PULSE METHOD OF FORMULATING A MATHEMATICAL
MODEL OF A PHYSICAL SYSTEM

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SUMMARY

Because of the increasing complexity of control systems and the more stringent requirements placed upon them, control engineers are turning more and more to the modern techniques of control system synthesis. Before these techniques can be applied, however, a mathematical model of the physical components is required. Unfortunately, the formulation of a mathematical model of a physical system or system component is essentially an art, rather than a completely logical procedure with specific rules. The specific approach often depends upon the background of the control engineer.

The purpose of this investigation is to experimentally and analytically evaluate a pulse method of formulating a mathematical model which describes the dynamic behavior of a physical system. The study will be limited to systems whose dynamics may be adequately described by an ordinary linear time-invariant differential equation of the form

$$\frac{d^3 r}{dt^3} + a_2 \frac{d^2 r}{dt^2} + a_1 \frac{dr}{dt} + a_0 r = He(t) \quad (1)$$

where $e(t)$ is the forcing function; $r(t)$ is the system response; and the a 's are constant coefficients.

The forcing function is to be a piecewise constant function given by

$$\begin{aligned} He(t) = P(kT) = P_k & \quad kT \leq t < (k+1)T \\ k = 0, 1, 2, \dots \end{aligned} \quad (2)$$

where the pulse width T is a positive constant assumed sufficiently short.

It is assumed that the system is initially in an equilibrium state such that

$$\frac{d^3 r}{dt^3} = \frac{d^2 r}{dt^2} = \frac{dr}{dt} = r = 0 \quad (3)$$

at $t = 0$. The experimental procedure consists of finding the minimal sequence of pulses $P(0)$, $P(T)$, $P(2T)$, ... capable of exciting the system and then returning it to the original equilibrium state simultaneously with the termination of the sequence.

The analytical study employs state space techniques to show the intimate relationship existing between the eigenvalues of the differential equation and the amplitudes of the pulses which achieve the prescribed response. A general procedure is formulated for determining the eigenvalues of the differential equation from the pulse heights.

The proposed method is used to identify a number of different systems simulated on the analog computer. Generally, the accuracy of the method seems superior to either the transient or frequency response methods when it is applicable.

CHAPTER I

INTRODUCTION

Background

In the early development of control systems, designers worked with the differential equations describing system behavior. The time domain techniques were unsatisfactory, however, because of the difficulty of relating an open loop parameter change to the closed loop transient response.

During the war years, the work of Bode, Hall, Nichols and others centered attention upon the frequency domain. Systems and system components were described by their steady state frequency responses. Design was carried out in the frequency domain working with the open loop frequency response. Bode's gain and phase margin, Nyquist's -1 point and Nichols' gain and phase charts gave the desired knowledge of the closed loop frequency characteristics from the open loop frequency characteristics. However, these methods had their shortcomings. Often, design specifications were given in the time domain as rise time, maximum overshoot, setting time, etc. The relationship between the steady state frequency response and the transient response is very complex if the system is at all complicated.

In the last decade, the work of Guillemin, Evans, Truxal and others has focused attention upon the complex frequency domain. Working in this domain, the designer has control over both the closed loop frequency and transient response.

Because of the increasing complexity of control systems and the more stringent requirements placed upon them, control engineers are turning more and more to the modern techniques of control system synthesis such as Evans' root locus technique and the Guillemin-Truxal method¹. Before these techniques can be applied, however, a mathematical model of the physical components is required. Unfortunately, the formulation of a mathematical model of a physical system has been somewhat neglected, and has not kept pace with the advances in synthesis techniques. The determination of a mathematical model of physical components aspect of control system design still remains essentially an art rather than a completely logical procedure. This is surprising because the success which can be expected from a specific design depends directly upon the precision with which the mathematical model describes the physical components.

The problem of determining a suitable mathematical model of a physical system will be called the "identification problem" to distinguish it from two closely related problems; those of analysis and synthesis. The three problems are interrelated and at times the dividing line is rather nebulous.

In general, the identification problem would include nonlinear and/or time-varying systems as well as linear and time-invariant systems. This study will consider a greatly reduced version of the general identification problem although the problem considered is sufficient to include the majority of the cases encountered in the design of control systems.

Definition of the Problem

We are given a physical system whose dynamics may be adequately described by an ordinary linear time-invariant differential equation of the form

$$\frac{d^3 r}{dt^3} + a_2 \frac{d^2 r}{dt^2} + a_1 \frac{dr}{dt} + a_0 r = H e(t) \quad (4)$$

where $e(t)$ is the forcing function

$r(t)$ is the system response

a_i 's are constant coefficients

H is a constant of proportionality.

We have complete control over the forcing function, and a record of the system response. The excitation and response may be variables in any physical system (electrical, mechanical, hydraulic, pneumatic, etc.) and need not be in the same units.* However, it is assumed that they are related in a linear time-invariant manner.

The problem is to determine a mathematical model which characterizes the dynamic relationship between the forcing function and the system response. The problem is considered solved when we can specify within a constant either the differential equation relating the system input to the output or the transfer function of the system defined as the ratio of the system response to the excitation, both expressed as a function of the complex frequency. Actually, the transient modes of the system are sufficient to identify the class of systems to be studied here within a

*The specific hardware developed during the thesis generates a voltage forcing function and requires a voltage response. If the system variables are other than voltage, suitable transducers would be required.

constant. Determination of the constant multiplier will not be discussed. It is frequently known from an understanding of the operation of the system. If not, it may often be determined by applying a unit step function excitation and noting that the steady-state response of Equation (4) is given by a_0/H .

Present Techniques of Linear System Identification

There are three general methods which may be used to identify linear systems.

Analytical analysis.--Through application of the fundamental laws of physics, a set of differential equations relating the various system variables may be written. The coefficients of the equations are evaluated individually either from manufacturers' data or by analytical or experimental means. The equations are then manipulated into a single equation relating the excitation to the response.

As a specific example, suppose the component to be identified is an electronic amplifier. An equivalent circuit for each stage would first be drawn and the tube characteristics then determined from a tube manual. Next, the value of each resistor, capacitor and inductor would be placed on the equivalent circuit. Then the gain, as a function of frequency would be calculated for each stage. Thus, a set of equations relating the input and output of each stage would be obtained. The intermediate variables would then be eliminated to arrive at a single equation relating the amplifier input to its output. A similar procedure would be carried out for each component of the system. The results of this method would be in the proper form for application of the modern

synthesis techniques. However, this method is not generally used in practice. Ordinarily, the time required to perform a detailed analysis of this type is considered prohibitive and a more direct method of identifying the system is sought. Also, because control engineering cuts across a large number of specialized engineering boundaries, the control engineer may not have sufficient background to perform a detailed analysis even if time were available. As a result of these difficulties, attention is usually focused upon those characteristics of the physical system which are known to be of importance rather than to attempt a complete determination of the various relationships existing between each of the system variables. Thus, the system is usually thought of as a "black box" and a dynamic relationship is sought between specific input and output variables. When this is done, the designer loses a detailed picture of the physics of operation of the process. It is no longer possible to estimate the effects of varying a specific parameter or to estimate the effect of corrupting signals entering the system at points other than the measured input upon the response. However, it is almost always possible to design a satisfactory control system knowing only the input-output relation for each component. Actually, the input-output relation for all the fixed components in cascade is all that is normally required. Thus, from this point on a physical system or system component will be thought of as a black box (two port transducer) with an input variable which may be controlled and a response variable which can be measured.

Frequency domain analysis.--A sinusoidal forcing function of constant amplitude is impressed upon the component input and the amplitude and

phase shift of the output is measured. This is done at a number of discrete frequencies through the frequency range of interest.

The results of the frequency response tests are usually plotted on semi-log paper as log magnitude ratio in decibels and phase shift in degrees versus a logarithmic frequency scale. A system description in this form is directly amenable to frequency-domain techniques. However, it is necessary to determine a transfer function which satisfies both the gain and phase characteristics before the modern synthesis techniques can be applied.

Time domain analysis.--An aperiodic forcing function is applied to the system input and the system response measured. Usually, either an impulse or step function is used to excite the system. This single measurement in the time domain is all that is necessary to uniquely characterize a linear time-invariant system. It contains all the information contained in the large number of measurements made in frequency domain. While the time domain measurement has clear advantages over the other two methods with respect to time required to perform the measurements, it also has disadvantages. It lacks the reinforcing abundance of information inherent in the numerous gain and phase measurements. Thus, it is more adversely affected by noise. Also, slight errors in the time domain measurement lead to significant changes in the mathematical model derived from the measurement. While slight errors in the frequency domain measurement will have relatively little effect on the resulting mathematical model, another major disadvantage of time domain measurement is the difficulty of determining the mathematical model from the measured data. Although there are a number of specialized

methods (the response of lightly damped second order systems can be compared with standard curves to determine the transfer function of the system), there is no straightforward general procedure for going from the excitation and response time representations to a mathematical model. The most general procedure is to first take the Fourier transformation of the time functions. The ratio of the response transformation to the excitation transformation is the transfer function evaluated at real frequencies. The Fourier transform can be accurately performed with the use of a digital computer. However, it is still necessary to determine the mathematical model from the frequency domain description. This involves another approximation which could conceivably magnify the error introduced in going from the time domain to the steady-state frequency domain. Thus, the ease with which the time domain measurement is made is certainly marred by the two-stage approximation and the extensive analysis required to arrive at the desired mathematical model.

One last point which should be mentioned is the difficulty of determining with some degree of accuracy the location of secondary modes when the system possesses several predominant modes. The predominant modes tend to mask the secondary modes from view. This is true to some extent whether the measurement is performed in the frequency or time domain. In the time domain, the secondary modes usually have much smaller residues than the more predominant modes. Thus, their contribution to the initial response is usually considerably less than the contribution of the predominant modes. Also, the portion of the transient response due to the secondary modes will die out much faster than that of the predominant modes. Thus, secondary modes are very difficult to detect from a standard ink recording of the impulse or step response of a physical system.

Because the high frequency modes usually contribute so little to the total system response, we are strongly motivated to neglect them. Whether these modes may be neglected or not depends upon the specific application. If they are included and not needed, they needlessly complicate the analysis. However, if these high frequency or secondary modes are neglected in the design of high performance control systems requiring very high loop gains, they may lead to unstable operation of the completed system. Thus, there are times when it is important to locate the secondary modes with some degree of accuracy.

The use of steady-state frequency response measurements does not completely circumvent the difficulty encountered in the time domain characterization. Usually, the break frequencies of the dominant modes are much lower than the break frequencies of the secondary modes. Each low frequency break point results in a twenty decibels per decade attenuation of the response at high frequencies (forty decibels per decade attenuation in the case of complex modes). Therefore, when performing frequency response tests, it is often found that the system response at the higher frequencies may become very small and difficult to distinguish from the noise level. This makes the high frequency break points very difficult to locate.

In the major portion of the work which has been done on the identification problem, the writers first assume that either a step, impulse, or steady-state frequency response test has been performed and then formulate a new procedure for determining the mathematical model from the results. Although the work in this direction has been tremendous, no simple accurate solution to either problem has been found.

It becomes evident that if the modern synthesis techniques are to be used to the fullest advantage, a more direct method of determining a mathematical model of physical systems and system components is necessary.

Recent work by Lendaris and Smith² has pointed out a novel approach for determining system pole locations. The system under test is excited by a "complex zero signal generator" and the response is observed on an oscilloscope. When a signal zero is moved to coincide with a system pole, the residue in that pole becomes zero. The effect of the cancelled pole can no longer be seen in the system response. Since the zero location is known, the pole location is thus determined.

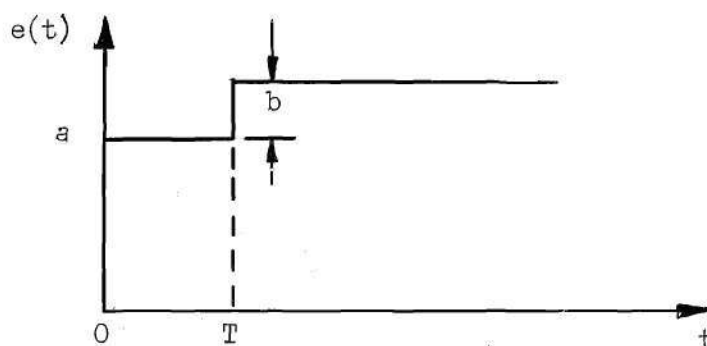


Figure 1. Signal Whose Laplace Transform Contains Complex Zeros.

A signal whose Laplace transform contains complex zeros is shown in the figure (1). Its equation is

$$e(t) = aU_{-1}(t) + bU_{-1}(t - T) \quad (5)$$

The Laplace transform of this equation is

$$E(s) = \frac{a}{s} + \frac{b}{s} e^{-sT} = \frac{1}{s} \left[a + be^{-sT} \right] \quad (6)$$

The Laplace transform of the signal has a pole at the origin and zeros where

$$a + be^{-sT} = 0 \quad (7)$$

or where

$$e^{-sT} = -\frac{a}{b} \quad (8)$$

By taking the natural logarithm of each side there results

$$-sT = \text{Ln} \left(\left| \frac{a}{b} \right| e^{j\phi} \right) \quad (9)$$

$$-sT = \text{Ln} \left| \frac{a}{b} \right| + j\phi \quad (10)$$

where

$$\phi = \begin{cases} (2k + 1)\pi & \text{if } b > 0 \end{cases}$$

$$\phi = \begin{cases} 2k\pi & \text{if } b < 0 \end{cases}$$

$$k = 0, 1, 2, \dots$$

Recalling that $s = \sigma + j\omega$ and equating the real and imaginary parts, we obtain

$$\sigma = -\frac{1}{T} \text{Ln} \left| \frac{b}{a} \right| \quad (11)$$

$$\omega = \frac{(2k + 1)\pi}{T} \quad \text{for } b > 0 \quad (12)$$

$$\omega = \pm \frac{2\pi k}{T} \quad \text{for } b < 0 \quad (13)$$

$$k = 0, 1, 2, \dots$$

Equations (11), (12) and (13) show that by varying b/a and T , it is possible to place a zero any place in the s -plane. Figure 2 shows typical pole-zero constellations for the possible double step function combinations.

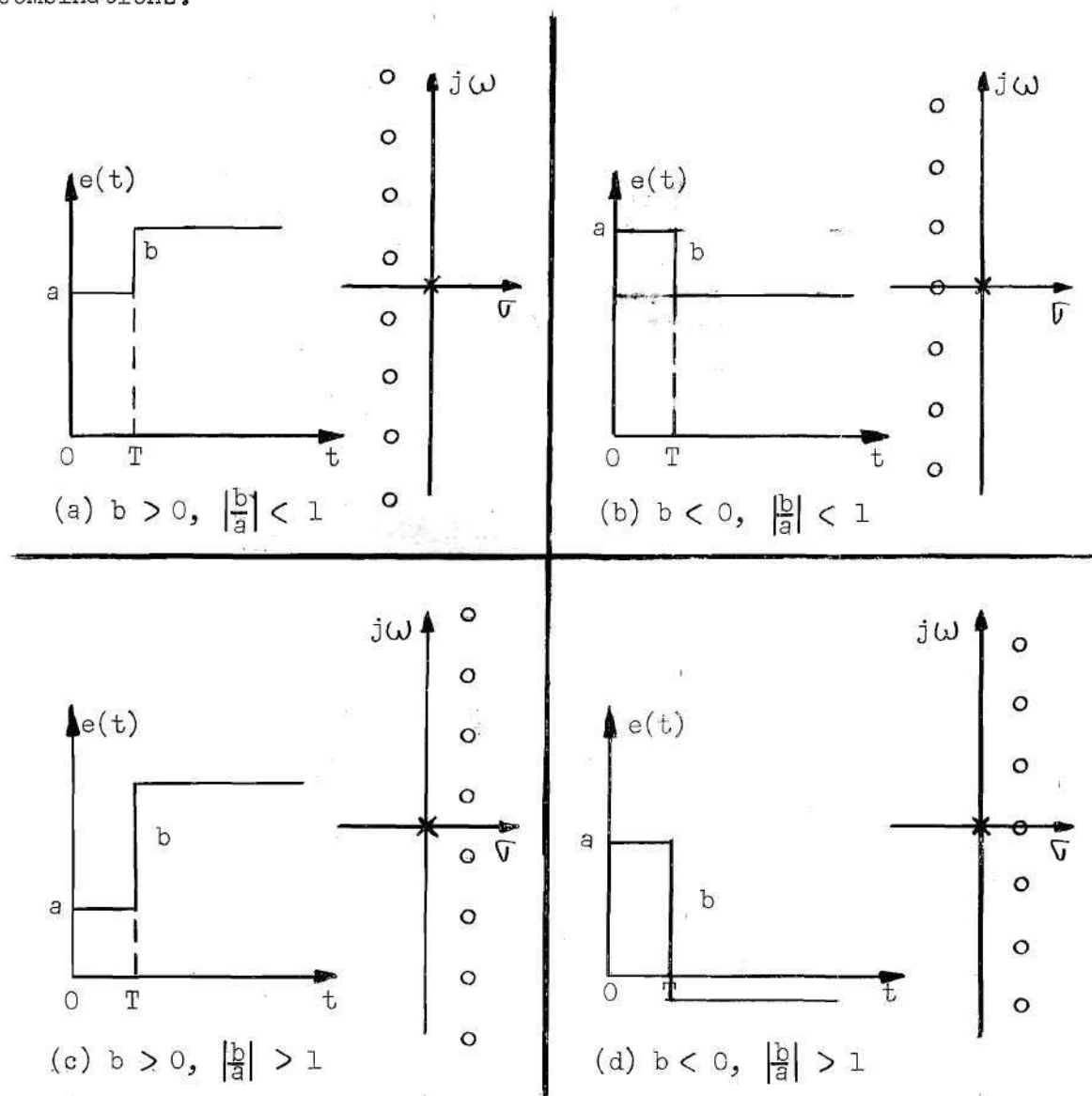


Figure 2. Typical Pole-Zero Constellations For Various Double Step Function Combinations.

The complex zero signal generator constructed by Lendaris and Smith consisted of two square wave generators and a variable time delay connected as shown in Figure 3.

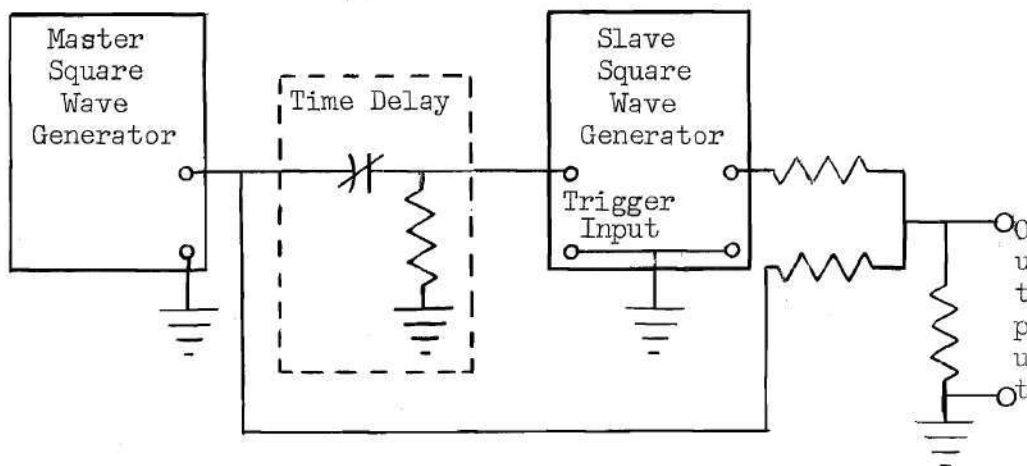


Figure 3. Complex Zero Signal Generator.

A master-square wave generator started the sequence. Its signal was applied to a resistative summing network and also to a variable time delay. The time delayed triggered the second square wave generator. The second square wave generator's output was also applied to the resistative summing network. The output of this network is connected to the input of the system to be tested. Brussolo³ applied this method to 35 different systems, each of which contained two pairs of complex poles. He concluded that in most instances the pole locations were determined in one-third of the time required by the frequency response method and with greater accuracy.

Unfortunately, the method does not aid in determining system zero locations and what is more important it is not applicable to systems which contain a pole at the origin. This eliminates practically all control systems from identification by this procedure. Actually, the

value of Lendaris and Smith's paper may not be so much in their method as in the implicit suggestion that it might be possible to perform a measurement, completely different from the step (or impulse) response test or the steady-state frequency response tests, that would materially reduce the computation required to process the test results into a mathematical model of the system and at the same time improve the accuracy of the model. It was with this thought in mind that this thesis was undertaken.

Proposed Method

The proposed method consists of repeatedly driving the system with a sequence of short pulses and observing the system response on an oscilloscope. The proposed driving function is defined by

$$e(t) = P(kT) = P_k \quad kT \leq t < (k+1)T$$

$$k = 0, 1, 2, \dots$$

where the pulse width T is a constant assumed to be sufficiently short. The amplitudes of the pulses, following the first pulse, are adjusted after each application of the pulse sequence in an effort to determine the minimal sequence capable of exciting the system and then returning it to its equilibrium state coincident with the completion of the pulse sequence.

Generally, when an aperiodic signal is passed through a linear system, the system response has an asymptote which extends to infinity. Thus, intuitively, one would expect that an aperiodic discontinuous pulse sequence which could pass through a linear system and leave it

unexcited would be related in some fundamental manner to the transient modes of the system. Actually, it may not be completely obvious that such a pulse sequence even exists whether related or unrelated to the transient modes, because the proposed forcing function terminates abruptly in a sharp discontinuity and yet, we have specified that the system response should remain zero after the completion of the pulse sequence.

It will be shown mathematically in the next two chapters that for the class of systems considered here, the pulse sequence does exist. The relationship between pulse amplitudes and the transient modes will be derived. Because the pulse method may be viewed as essentially a time domain method, the analysis to follow will be conducted directly in the time domain.

CHAPTER II

LINEAR SYSTEMS FROM THE STATE SPACE POINT OF VIEW

The recently developed "state" space concept is perhaps the most natural and most powerful method to study the behavior of dynamical systems directly in the time domain. It represents a uniting of the phase plane of Poincaré⁴, and classical theory of differential equations, and the theory of linear algebra. The result is a very powerful tool indeed. It has been recently used by LaSalle⁵ in solving the bang-bang control problem, and by Kalman and Bucy⁶ in their contribution to the theory of linear filtering and prediction. The most general method known for the study of stability, the "second method" of Lyapunov⁷, requires the use of the "state" concept.

From the state space point of view, a physical system is represented by a vector in n -dimensional space called the state vector. The state vector completely describes the behavior of the physical system. When the response of the physical system changes, the state vector changes in a corresponding manner in state space. The classical definition of a vector is a quantity which has both magnitude and direction. An equally satisfying description could be achieved by the terminal point of a vector of proper magnitude and direction emanating from the origin of a co-ordinate system. If the vector is to exist in a space of more than three dimensions and, thus, have more than three components, the concept of magnitude and direction of a vector becomes difficult to visualize. For this reason, in the theory of modern algebra, a vector is thought of

as a point in n -dimensional space, and defined as an ordered array of numbers. An n -component vector is an ordered n -tuple of numbers. Thus,

$$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is an n -component vector. The v_i , $i = 1, \dots, n$ are called components of the vector. To study the behavior of a physical system from the state space point of view, it is necessary to describe the system with a set of first order differential equations.

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \quad (14)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t)$$

.....

$$\frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t)$$

All physical systems which can be described by an ordinary differential equation or a system of ordinary differential equations can, in general, be reduced to this form⁸. In fact, this is often referred to as the general form. These equations may be concisely expressed:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \quad (15)$$

$$i = 1, 2, \dots, n$$

This equation, in vector form, becomes the vector differential equation:

$$\frac{d\bar{x}(t)}{dt} = f(\bar{x}(t), \bar{u}(t), t) \quad -\infty \leq t < \infty \quad (16)$$

The vector $\bar{x}(t)$ is the state vector. Its components $x_i(t)$ are the state variables. The state of a dynamic system is the smallest collection of numbers which must be specified at time $t = t_0$ in order to be able to predict the behavior of the free system for any time $t \geq t_0$. The vector $\bar{u}(t)$ is the vector description of the forcing function. The physical system is described by the vector valued function f . The integer n is the order of the system. If $\bar{u}(t) = \bar{0}$ for all t , the system is said to be free or unforced.

$$\frac{d\bar{x}(t)}{dt} = f(\bar{x}(t), t) \quad -\infty < t < \infty \quad (17)$$

A state $\bar{x}_e(t)$ of a free dynamical system is an equilibrium state if

$$f(\bar{x}_e(t); t) = 0 \quad \text{for all } t \quad (18)$$

In other words, a motion passing through an equilibrium state at any time is actually at the same state at all times.

A dynamic system Equation (16), is stationary if

$$f(\bar{x}(t), \bar{u}(t), t) = f(\bar{x}(t), \bar{u}(t)) \quad \text{for all } \bar{x}(t), \bar{u}(t) \quad (19)$$

Therefore, if f does not depend explicitly on time, the system is stationary. Thus, for a free system, every motion is invariant under transition in time. A system which is both free and stationary is said to be autonomous.

A dynamic system is linear if f is a linear function of $\bar{x}(t)$ and $\bar{u}(t)$. If it were assumed for simplicity that $\bar{0}$ is an equilibrium state, i.e.,

$$f(\bar{0}, \bar{0}, t) = \bar{0} \quad , \quad (20)$$

then for a linear system, Equation (16) may be written in the form

$$\frac{d\bar{x}(t)}{dt} = A\bar{x}(t) + P(t) \bar{b} \quad , \quad (21)$$

where

$\bar{x}(t)$ = "state" of the system at time t

A = $n \times n$ matrix describing the system

$P(t) \bar{b} = \bar{u}(t)$ = vector description of the forcing function

(assuming the system has a single input, $P(t)$).

Equation (21) has the well known solution⁹

$$\bar{x}(t) = G(t) \bar{x}(0) + G(t) \int_0^t G(t')^{-1} P(t') \bar{b} dt' \quad (22)$$

The matrix $G(t)$ is referred to by several names. Mathematicians often call it the principal fundamental matrix. Engineers usually refer to it either as the impulse response matrix or the transition matrix. The transition matrix satisfies its own differential equation

$$\frac{dG(t)}{dt} = A G(t), \quad G(0) = I \quad (23)$$

where I is the unit matrix. Restricting the system to be time-invariant makes A and \bar{b} constant matrices. The transition matrix then becomes the matrix exponential

$$G(t) = I + At + \frac{(At)^2}{2!} + \dots = e^{At} \quad (24)$$

Some important properties of the transition matrix due to its exponential nature follow.

$$G(t) G(t') = G(t + t') \quad (25)$$

Letting $t' = -t$

$$G(t) G(-t) = I \quad (26)$$

but

$$G(t) G(t)^{-1} = I \quad (27)$$

therefore

$$G(-t) = G(t)^{-1} \quad (28)$$

Also from Equation (25)

$$G(t)^n = G(nt) \quad (29)$$

These properties of $G(t)$ will be of later use. It might be noted that they are invariant under a colinear transformation. Now, applying

Equations (25) and (28) to Equation (22), it becomes

$$\bar{x}(t) = G(t) \bar{x}(0) + \int_0^t G(t - t') P(t') \bar{b} dt' \quad . \quad (30)$$

The solution of Equation (21) for the constant coefficient case is the product of the impulse response matrix and the initial conditions plus the forced solution which is the convolution of the forcing function with the impulse response matrix. Since the forcing function is to be a piecewise constant function

$$P(t) = P(kT) = P_k \quad kT \leq t < (k+1)T \quad . \quad (31)$$

It will be advantageous to use the discrete analogue of Equation (30)

$$\bar{x}(k+1)T = G(T) \bar{x}(kT) + P_k \int_0^T G(t - t') \bar{b} dt' \quad (32)$$

$$kT \leq t < (k+1)T$$

Equation (32) is a vector difference equation which gives the state of the system at discrete instances of time $T, 2T, \dots, kT, \dots$

Defining

$$\bar{h}(T) = \int_0^T G(t - t') \bar{b} dt' \quad , \quad (33)$$

Equation (32) becomes

$$\bar{x}(k+1)T = G(T) \bar{x}(kT) + P_k \bar{h}(T)$$

$$kT \leq t < (k+1)T \quad .$$

In the event we wish to determine the behavior of the system over a specific interval, we may use a modified form of Equation (33)

$$\bar{x}((k + t')T) = G(t') \bar{x}(kT) + P_k \bar{h}(t') \quad (34)$$

$$kT \leq t' < (k + 1)T$$

We are now in a position to consider a specific example. We will first assume that we know the form of the differential equation which describes the system and solve for the pulse heights which force the system to yield the prescribed response. The first example will be a first order system. Although a first order differential equation is too simple to describe many physical systems, this example will later be of aid in visualizing the response of higher order systems.

Example of a first order system:

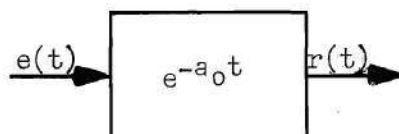


Figure 4. Linear First Order System

The system is described by the differential equation

$$\frac{dr(t)}{dt} = -a_0 r(t) + H e(t) \quad (35)$$

The discrete solution of Equation (35) is

$$r((k + 1)T) = e^{-a_0 T} r(kT) + P_k h(T) \quad (36)$$

where

$$h(T) = \frac{1 - e^{-a_0 T}}{a_0}$$

and

$$P(kT) = P_k = H e(t) \quad k \leq t < (k+1)T \quad . \quad (37)$$

If the system is in its free equilibrium state at $t = 0$, the state of the system at the completion of the first pulse ($t = T$) is

$$r(T) = P_0 h(T) \quad . \quad (38)$$

At the completion of the second pulse, the state of the system is given by

$$r(2T) = e^{-a_0 T} P_0 h(T) + P_1 h(T) \quad . \quad (39)$$

When the second pulse is properly chosen, the system will be returned to its original equilibrium state at its completion. Thus, $r(2T) = 0$ and $h(T)$ may be factored out because $h(T)$ cannot be zero. Equation (39) becomes

$$e^{-a_0 T} P_0 + P_1 = 0 \quad , \quad (40)$$

or

$$\frac{P_1}{P_0} = -e^{-a_0 T} \quad . \quad (41)$$

Assuming P_0 is a positive pulse, then P_1 must be negative.

We may solve Equation (41) for a_0 in terms of the pulse heights

$$a_0 = -\frac{1}{T} \ln\left(-\frac{P_1}{P_0}\right) \quad . \quad (42)$$

A first order system may be described in a state space of one dimension. Thus, the movement of a point along a line completely describes the state of the system. Time does not appear explicitly in state space but appears implicitly along the trajectories of the point. A plot of the position of the point versus time would have the form shown in Figure 5 when the second pulse is correctly adjusted.

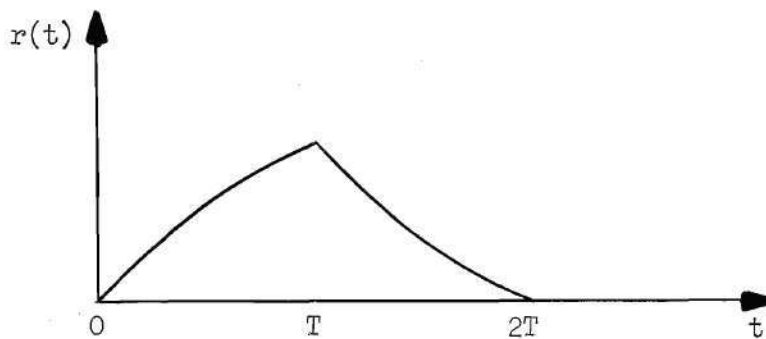


Figure 5. Response of First Order System
When Pulses Are Properly Chosen

In Figure 5, the pulse width T is equal to the time constant of the system or $Ta_0 = 1$. Of course, this is not necessary. However, it should be noted that as Ta_0 increases, the height of the second pulse decreases and becomes more difficult to measure accurately. Thus, for practical reasons, the pulse width should not greatly exceed the time constant of the system. If the height of the second pulse is improperly adjusted, the system will not be returned to its equilibrium state at the completion of the second pulse. Assuming no additional pulses are applied to the system, it will then approach equilibrium with its own free motion.

Figure 6 shows the response when the second pulse is too small. Likewise, if the magnitude of the second pulse is too large, the system will be left in a negative state and will approach equilibrium in an exponential manner from a negative direction.

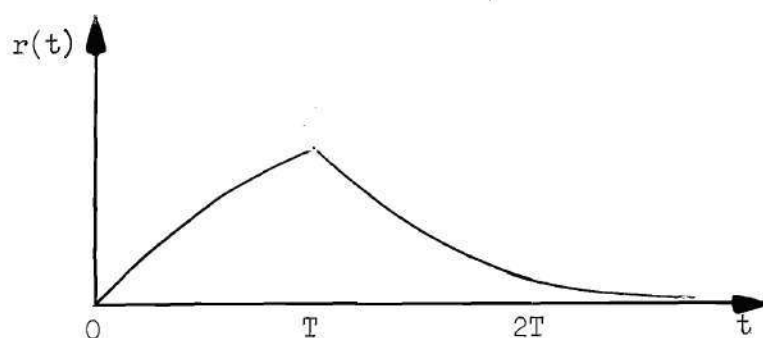


Figure 6. Response of First Order System When The Magnitude of The Second Pulse Is Too Small

Thus, at least for a first order system, we have a clear indication of how the second pulse amplitude should be adjusted to drive the system to the equilibrium state precisely at the termination of the second pulse. Once the second pulse is properly adjusted, the heights of both pulses are measured and used in Equation (42) along with the pulse widths to compute the coefficient of the differential equation. At least for a first order system, this procedure has a clear advantage over either the frequency response method or the time response method.

Example of a second order system:

$$\frac{d^2 r(t)}{dt^2} + a_1 \frac{dr(t)}{dt} + a_0 r(t) = H e(t) \quad (43)$$

To study the system in state space

let

$$x_1 = r(t) \quad (44)$$

and

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -a_0 x_1 - a_1 x_2 + He(t)$$

using vector notation

$$\frac{d\bar{x}}{dt} = A\bar{x} + P(t)\bar{b} \quad , \quad (45)$$

where

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad P(t)\bar{b} = He(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The discrete solution of Equation (45) is

$$\begin{aligned} \bar{x}((k+1)T) &= G(T)\bar{x}(kT) + P_k \bar{h}(T) \quad , \\ kT \leq t &< (k+1)T \quad . \end{aligned} \quad (46)$$

Although a number of methods are given in the literature for determining the transition matrix,^{10, 11, 12} the method used here is slightly different from any of them. It has the advantage of circumventing the difficulties which generally occur when the system has multiple roots and it utilizes Laplace transforms which are more familiar to most engineers than the classical theory of differential equations. Also, the signal flow graph will place in evidence the interrelationships existing among the state variables that are obscured by the compact matrix notation. The theory of Laplace transforms and signal flow graphs are well documented in the literature and will not be discussed here. To evaluate the transition matrix, first draw a signal flow graph of Equation (44).

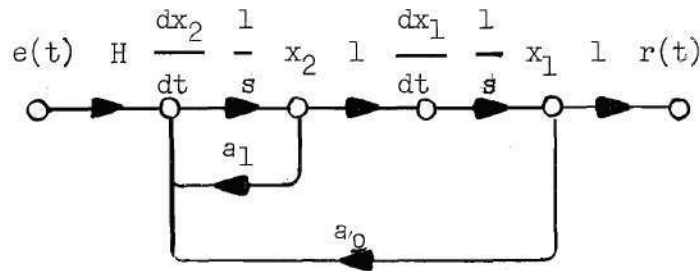


Figure 7. Signal Flow Graph of Second Order System

The nodes of the signal flow graph represent the system variables. The unilateral branches display the interrelationships existing among the variables. The transition matrix may be determined from the flow graph by noting

$$G(t) = \mathcal{L}^{-1} \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \quad (47)$$

that each element, $g_{ij}(s)$, of the transition matrix is given by the graph transmittance from the $\frac{dx_j}{dt}$ node to the x_i node. The convenience of using signal flow graphs is enhanced because Mason¹³ has derived a general equation for the graph transmittance

$$T(s) = \frac{1}{\Delta(s)} \sum_k t_k(s) \Delta_k(s) \quad (48)$$

where

$T(s)$ = source-to-sink graph transmittance

$t_k(s)$ = transmission of the k^{th} source to sink path

$\Delta(s)$ = graph determinant

$\Delta_k(s)$ = determinant of that part of the graph not touching the k^{th} path.

Using Equation (48), the transition matrix for Equation (44) becomes

$$G(t) = e^{-1} \begin{bmatrix} \frac{s + a_1}{s^2 + a_1 s + a_0} & \frac{1}{s^2 + a_1 s + a_0} \\ \frac{-a_0}{s^2 + a_1 s + a_0} & \frac{s}{s^2 + a_1 s + a_0} \end{bmatrix} . \quad (49)$$

Recalling

$$\bar{h}(t) = \int_0^t G(t - t') \bar{b} dt \text{ and } \bar{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

it may be seen that $\bar{h}(t)$ is the integral of the second column of the transition matrix,

$$\bar{h}(t) = e^{-1} \begin{bmatrix} \frac{1}{s(s^2 + a_1 s + a_0)} \\ \frac{1}{s^2 + a_1 s + a_0} \end{bmatrix} . \quad (50)$$

To evaluate the inverse transform, it is necessary to factor the characteristic equation

$$s^2 + a_1 s + a_0 = 0 , \quad (51)$$

which gives

$$s = -a_1/2 \pm \left[(a_1/2)^2 - a_0 \right]^{1/2} . \quad (52)$$

There are three possible results. They are

$$(a_1/2)^2 > a_0, \quad (a_1/2)^2 < a_0 \quad \text{or} \quad (a_1/2)^2 = a_0 .$$

Case I: $(a_1/2)^2 > a_0$

let

$$\sigma = a_1/2$$

and

$$\omega_r = \left[(a_1/2)^2 - a_0 \right]^{\frac{1}{2}} .$$

Then

$$s_1 = -\sigma \pm \omega_r . \quad (53)$$

The transition matrix is given by

$$G(T) = e^{-\sigma T} \begin{bmatrix} \cosh \omega_r T + \frac{\sigma}{\omega_r} \sinh \omega_r T & \frac{1}{\omega_r} \sinh \omega_r T \\ -\frac{a_0}{\omega_r} \sinh \omega_r T & \cosh \omega_r T - \frac{\sigma}{\omega_r} \sinh \omega_r T \end{bmatrix} . \quad (54)$$

Let the system be in the equilibrium state

$$\frac{d\bar{x}_e}{dt} = \bar{x}_e = \bar{0} \quad \text{at } t = 0 . \quad (55)$$

The state of the system at the completion of the first pulse, $t = T$, becomes

$$\bar{x}(T) = P_0 \bar{h}(T) , \quad (56)$$

and at the completion of the second pulse, the state of the system is

$$\bar{x}(2T) = G(T) P_0 \bar{h}(T) + P_1 \bar{h}(T) \quad . \quad (57)$$

At the completion of the third pulse the state of the system is given by

$$\bar{x}(3T) = G(T) \left[G(T) P_0 \bar{h}(T) + P_1 \bar{h}(T) \right] + P_2 \bar{h}(T) \quad . \quad (58)$$

It will be shown in Chapter III that a sequence of three pulses is all that is necessary to excite a second order system and then return it to its equilibrium state. Thus, when the second and third pulses are properly adjusted, the system will be returned to its equilibrium state at the completion of the pulse sequence.

$$\bar{x}(3T) = \bar{0} = G(T)^2 P_0 \bar{h}(T) + G(T) P_1 \bar{h}(T) + P_2 \bar{h}(T) \quad (59)$$

Factoring out P_0 and $\bar{h}(T)$ neither of which are zero, gives

$$G(2T) + G(T) \frac{P_1}{P_0} + \frac{P_2}{P_0} I = \bar{0} \quad (60)$$

where I is the unity matrix.

In explicit notation, Equation (60) becomes

$$e^{-2\sigma T} \left(\cosh 2\omega_r T + \frac{1 + \sigma}{\omega_r} \sinh 2\omega_r T \right) \quad (61)$$

$$+ e^{-\sigma T} \left(\cosh \omega_r T + \frac{1 + \sigma}{\omega_r} \sinh \omega_r T \right) \frac{P_1}{P_0} + \frac{P_2}{P_0} = 0$$

$$e^{-2\sigma T} \left(\cosh 2\omega_r T - \frac{a_0 + \sigma}{\omega_r} \sinh 2\omega_r T \right) \quad (62)$$

$$+e^{-\sigma T} \left(\cosh \omega_r T - \frac{a_0 + \sigma}{\omega_r} \sinh \omega_r T \right) \frac{P_1}{P_0} + \frac{P_2}{P_0} = 0$$

Solving Equations (61) and (62) for the pulse heights yields

$$\frac{P_1}{P_0} = -2e^{-\sigma T} \cosh \omega_r T \quad (63)$$

$$\frac{P_2}{P_0} = e^{-2\sigma T} \quad (64)$$

Equations (63) and (64) may be solved for σ and ω_r in terms of the pulse heights.

This gives

$$\omega_r = \frac{1}{T} \cosh^{-1} \frac{-P_1}{2(P_0 P_2)^{\frac{1}{2}}} \quad (65)$$

and

$$\sigma = -\frac{1}{2T} \ln \frac{P_2}{P_0} \quad (66)$$

Case II: $(a_1/2)^2 < a_0$

$$s_1 = -\sigma + j\omega_r \quad (67)$$

We could proceed in a manner similar to Case I, but by substituting $j\omega_r$ for ω_r in Equation (63) and noting that

$$\cosh j\theta = \cos \theta$$

we obtain

$$\frac{P_1}{P_0} = -2e^{-\sigma T} \cos \omega_r T \quad (68)$$

$$\frac{P_2}{P_0} = e^{-2\sigma T} \quad (69)$$

Solving Equations (68) and (69) for σ and ω_r in terms of the pulse heights results in

$$\omega_r = \pm \frac{1}{T} \cos^{-1} \frac{-P_1}{2(P_0 P_2)^{\frac{1}{2}}} \quad (70)$$

$$\sigma = -\frac{1}{2T} \ln \frac{P_2}{P_0} \quad (71)$$

Actually, in solving for the coefficients in terms of pulse heights, we have neglected to mention that the logarithm is a multiple valued function with infinitely many values. As long as we know the system contained only real poles, the use of the principal value was correct. However, in the case being considered, the system has complex poles and the multiple values take on a new meaning. To be completely correct, Equation (70) should be written

$$\omega_r = \pm \frac{1}{T} \cos^{-1} \frac{-P_1}{2(P_0 P_2)^{\frac{1}{2}}} \pm 2\pi m \quad m = 0, 1, 2, \dots \quad (72)$$

where

$$0 \leq \frac{1}{T} \cos^{-1} \frac{-P_1}{2(P_0 P_2)^{\frac{1}{2}}} \leq \pi \quad (73)$$

is the principal value. Of course, we would like to use only the principal value. To do this, we should choose our pulse widths such that

$$\omega_r T < \pi \quad (74)$$

or

$$T < \frac{1}{2f_r} \quad \text{where } \omega_r = 2\pi f_r \quad (75)$$

Until now, we have stated that the pulses should be sufficiently short without stating just how short sufficiently short might be. Now Equation (75) gives us a quantitative value which the pulse widths should not exceed. It may seem presumptuous to say that the pulse width should be less than some quantity which we are attempting to determine. However, this is exactly what is done when performing an impulse response test, where the rule of thumb is, the pulse width should be less than one-fourth the minimum significant time constant of the system.¹⁴

Case III: $(a_1/2)^2 = a_0$. Therefore, $\omega_r = 0$

and

$$s_i = \frac{a_1}{2} = \sigma \quad (76)$$

We could, of course, begin as we did in Case I. However, to conserve space let us again use Equation (63) and this time take the limit as ω_r approaches zero.

$$\lim_{\omega_r \rightarrow 0} 2e^{-\sigma T} \cosh \omega_r T = 2e^{-\sigma T} \quad (77)$$

Hence, for the case of repeated roots,

$$\frac{P_1}{P_0} = -2e^{-\sigma T} \quad (78)$$

and

$$\frac{P_2}{P_0} = e^{-2\sigma T} \quad (79)$$

Equations (78) and (79) may be solved for the repeated root in terms of the pulse heights

$$\sigma = -\frac{1}{T} \ln \frac{-P_1}{2P_0} \quad (80)$$

or

$$\sigma = -\frac{1}{2T} \ln \frac{P_2}{P_0} \quad (81)$$

At least for a first or second order system of the class specified in Chapter I, we see that the pulse amplitudes necessary to achieve the prescribed response are uniquely determined by the system. Unfortunately, the converse is not true. The system is not uniquely determined by the minimal pulse sequence. However, if the pulse widths are required to satisfy Equation (75) we can uniquely determine the system by using only the principal value of the multiple valued function.

One advantage of the state space representation is the geometric interpretation which exists when the system is of third order or less. The second order system of our example may be uniquely characterized in a state space of two dimensions. If the system response and its first derivative are used as the co-ordinates, the plot is often referred to as the phase plane. Figure 8a shows the step response of a second order system for different values of $\sigma/(a_0)^{\frac{1}{2}}$. The system is in its free equilibrium state at the time the step function is applied. The phase plane trajectories of a second order system with a properly adjusted pulse sequence forcing function is shown in Figure 8b. At the completion of the pulse sequence, both the response and its derivative are zero. For an n^{th} order system, it is necessary that the response and its $n-1$ derivatives be made zero simultaneously at the completion of the pulse sequence.

In this chapter, the order of the differential equation and the nature of the eigenvalues were assumed known. Then the pulse amplitudes were found. It was shown that the pulse amplitudes were related to the eigenvalues of the differential equation and, if numerical values for the pulse amplitudes were known, the eigenvalues could be computed. For the method to be of value, however, we should be able to determine the eigenvalues and, thus, the differential equation without a priori knowledge as to the order of the differential equation or the nature of its eigenvalues. This will be discussed in the next chapter.

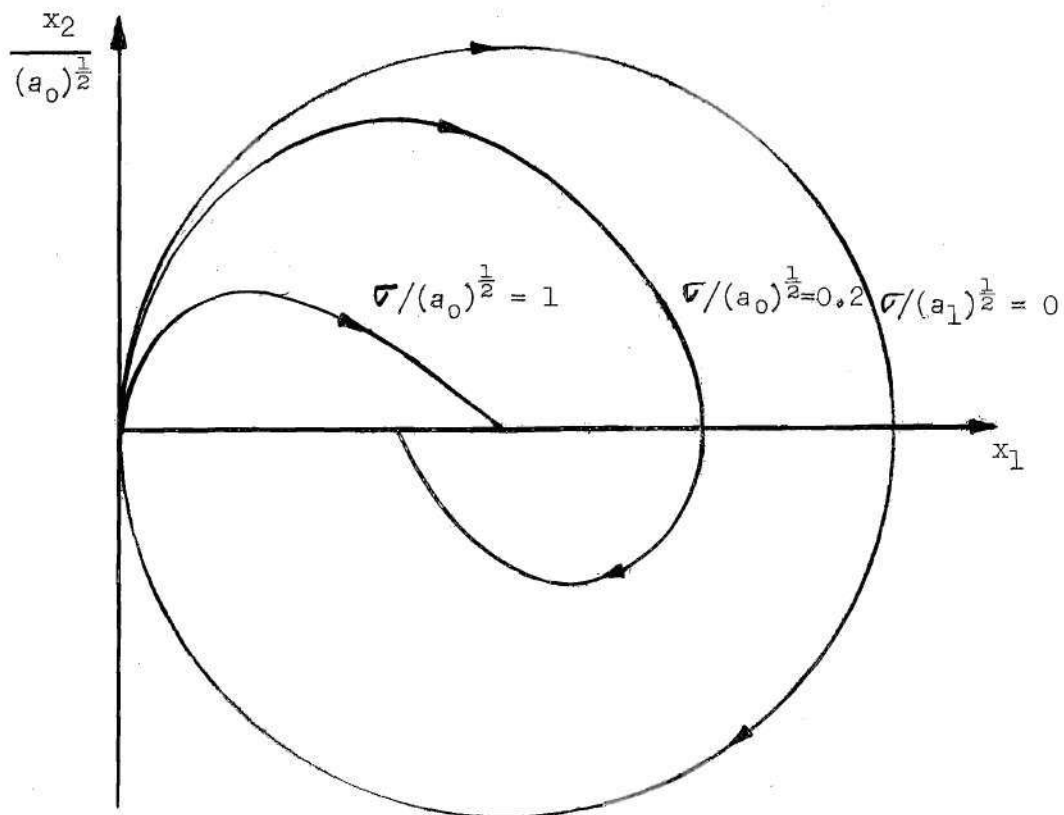


Figure 8a. State Space Trajectories of a Second Order System With a Step Forcing Function

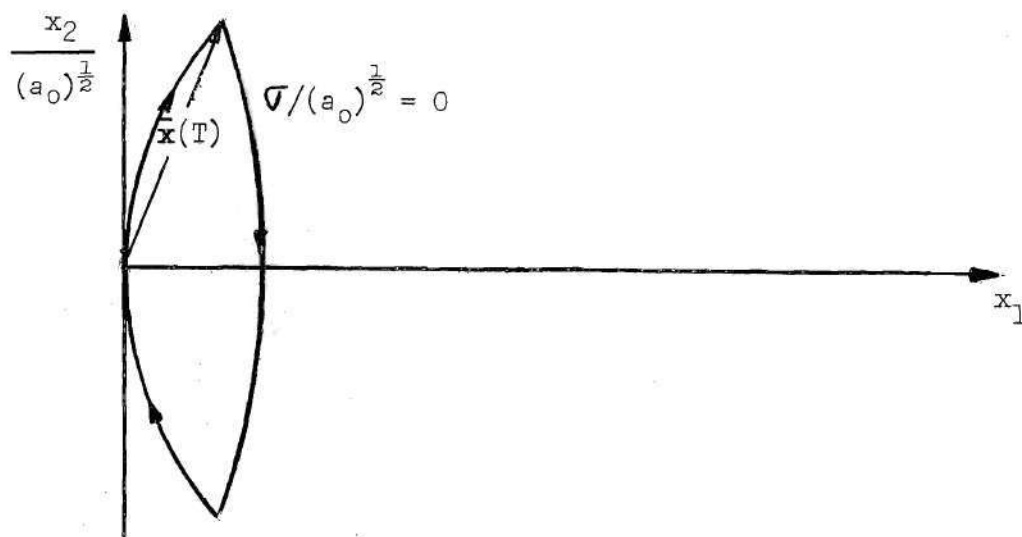


Figure 8b. State Space Trajectories of a Second Order System With a Properly Adjusted Pulse Sequence Forcing Function

CHAPTER III

THE RELATIONSHIP BETWEEN THE PULSE AMPLITUDES
AND THE SYSTEM EIGENVALUES

In Chapter II, examples were given showing that the pulse sequence which could pass through a system and leave it unexcited was related to the eigenvalues of the system. The examples assumed a priori knowledge of the number and nature of the eigenvalues. It is important, however, that we be able to determine the system eigenvalues from the pulse sequence that forces the system to yield the prescribed response without advance knowledge as to the number or nature of the eigenvalues. This is the purpose of this chapter. Assume the system may be adequately characterized by an ordinary linear time-invariant differential equation of the form

$$\frac{d^n r}{dt^n} + a_{n-1} \frac{d^{n-1} r}{dt^{n-1}} + \dots + a_1 \frac{dr}{dt} + a_0 r = H e(t) \quad (82)$$

where $e(t)$ is the forcing function

$r(t)$ is the system response

a_i $i = 0, 1, \dots, n-1$ are constant coefficients

H is a constant of proportionality.

To study this system from the state space point of view
let

$$x_1 = r(t)$$

and

$$\frac{dx_1}{dt} = x_2 \quad (83)$$

$$\frac{dx_2}{dt} = x_3$$

.

.

.

$$\frac{dx_n}{dt} = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + H e(t)$$

In vector notation, Equation (83) becomes

$$\frac{d\bar{x}(t)}{dt} = A \bar{x}(t) + P(t) \bar{b} \quad (84)$$

where

$$\bar{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \quad \text{and } P(t) \bar{b} = H e(t) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

We will make the additional assumption that the eigenvalues of A are distinct. This assumption is made because of mathematical complications that may occur when there are multiple eigenvalues. The assumption is actually not as restrictive as it may sound. That this is so may be stated as a theorem.¹⁵

Given any matrix A , there exists a matrix B such that $|A - B| < \epsilon$, where ϵ is any preassigned positive quantity. In other words, if a physical system having multiple roots is described exactly by a matrix A , the system may be approximately described by a matrix B having only distinct roots and the error involved can be made arbitrarily small. Actually, our final result will be valid regardless of the nature of the characteristic roots but this assumption will greatly simplify the mathematics.

Recognizing that $\bar{x}(t)$ can be treated as a time-dependent vector in n -dimensional space permits us to perform linear transformations upon $\bar{x}(t)$. There are several interpretations which may be given to a linear transformation. Often, it is thought of as a one-to-one mapping of the n -dimensional space onto itself in which every vector \bar{v} is uniquely mapped onto a vector \bar{v}' .¹⁶ The null vector is always mapped onto itself. Thus, the origin of the co-ordinate system remains unchanged. An equally valid interpretation of the transformation is that the vector \bar{v} remains unchanged but the frame of reference is changed. The origin, of course, remains unchanged as before. In other words, a linear transformation may be represented as a transformation from one point in space to another with respect to the same reference system or it may be represented as a change from one co-ordinate system to another, the vector remaining unaltered. The latter interpretation will be the most convenient for our purpose. A particularly useful co-ordinate system, which allows us to focus our attention on the fundamental characteristics of the system is called the normal co-ordinate system. When transformed into normal co-ordinates, the set of n simultaneous differential Equations (83) become n distinct differential equations, the solutions

of which are easily determined. Equation (84) may be transformed into normal co-ordinates in the following manner:

Let

$$\bar{x}(t) = T \bar{y}(t) \quad (85)$$

where T is an $n \times n$ non-singular matrix to be specified later.

$\bar{y}(t)$ and $\bar{x}(t)$ represent the same state vector. $\bar{y}(t)$ is referred to the new co-ordinate system.

Equation (84) becomes

$$T \frac{d\bar{y}(t)}{dt} = A T \bar{y}(t) + P(t) \bar{b} \quad (86)$$

premultiplying by T^{-1}

$$\frac{d\bar{y}(t)}{dt} = T^{-1} A T \bar{y}(t) + P(t) T^{-1} \bar{b} \quad (87)$$

We would like to choose T so that

$$T^{-1} A T = D = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \circ & \\ & & & \ddots \\ \circ & & & & s_n \end{bmatrix} \quad (88)$$

where s_i $i = 1, 2, \dots, n$ are the eigenvalues of A .

The existence of a matrix T^* such that Equation (88) is satisfied

*The procedure for determining the T matrix is given in Appendix A.

is assured by the stipulation that A have distinct eigenvalues.¹⁸ Thus, in normal co-ordinates, Equation (84) becomes

$$\frac{d\tilde{y}(t)}{dt} = D \tilde{y}(t) + P(t) \tilde{g} \quad (89)$$

where

$$\tilde{g} = T^{-1} \bar{b} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

The matrices A and D are said to be similar. Thus, they have equal determinants, the same characteristic equations, and the same eigenvalues. Once we accept the fact that Equations (84) and (89) characterize the same system, define the same vector in state space, and have identical characteristic roots, we may work with the normal co-ordinate equation and our analysis will be greatly simplified. The transformation into normal co-ordinates performs what is sometimes spoken of as an uncoupling of transient modes.¹⁹ In place of our original n^{th} order differential equation, we now have n independent first order differential equations. This makes it possible to study the effect of the driving function on each mode individually. At any time, the state of the system can be thought of as being the superposition of the individual modes of oscillation. Example 1 in the appendix gives a specific example of a third order system in normal co-ordinate state space.

In state space, we now have n new co-ordinates defined by n vectors. The vectors are actually the eigenvectors of the A matrix. Since the eigenvalues of the A matrix were distinct, the vectors will be linearly independent, and n linearly independent vectors form a basis in n -dimensional space. Thus, it is possible to represent any vector in state space as a linear combination of the n -vectors.

Equation (89) is equivalent to the n scalar first order differential equations

$$\frac{dy_i(t)}{dt} = s_i y_i(t) + P(t) g_i \quad i = 1, 2, \dots, n \quad (90)$$

In Chapter II, we discussed a first order differential equation and said it could be uniquely defined in a space of one dimension, i.e., by a point moving along a line. For a moment, let us assume the eigenvalues of A are all real. Then, each of the n Equations (90) may be described by a point moving along a vector in n -dimensional space. The vectors are the eigenvectors of the A matrix which form the normal co-ordinates. The state of the system at any time is given by the vector sum of the n vectors defined by the n points. Each point moves along its vector with a law of motion governed by the corresponding first order differential equation. When the system is unforced, each point moves with a motion of the form

$$k e^{s_i t}$$

When the system is in the equilibrium state

$$\frac{d\bar{y}(t)}{dt} = \bar{y}(t) = \bar{0}$$

each point is at the origin. If, at this time, a pulse is applied to the system, the points will fan out from the origin in n different directions, each moving at a different rate. If, at the termination of the initial pulse, another pulse of opposite polarity is applied to the system, the points immediately reverse their direction and start back toward the origin.

In Chapter II, it was shown that two pulses could excite a first order system and then return it to its equilibrium state exactly at the termination of the second pulse. The second pulse, however, was uniquely determined by the first pulse height, the pulse width, and the eigenvalue of the system,

$$P_1 = -P_0 e^{s_1 T} \quad (91)$$

Thus, the second pulse may be chosen to bring any one, but only one, of the points to the origin coincident with its termination. If no additional pulses are applied, the remaining $(n - 1)$ points will move toward the origin with their own free motion. There are times when this procedure might be used to locate a dominant eigenvalue or remove it from the response so that the less prevalent modes might better be observed. In Appendix A, an example is worked out which should clarify any obscure points in the preceding discussion.

In general, we would not expect the system to have all real eigenvalues. However, since the A matrix is real, complex eigenvalues will occur in conjugate pairs and have complex conjugate eigenvectors. Although it is permissible to speak of complex vectors in vector space, it will always be possible to combine them so that the motion due to complex eigenvalues takes place in a plane.²⁰ When the system is free,

the motion will have the form of an exponential spiral.

To determine the eigenvalues from the pulses, let us solve the vector differential Equation (89) which is equivalent to the n scalar differential equations

$$\frac{dy_i(t)}{dt} = s_i y_i(t) + P(t) g_i \quad i = 1, 2, \dots, n \quad (90)$$

By multiplying by the integrating factor, $e^{-s_i t}$, and rearranging, the equations become

$$\frac{d}{dt} (e^{-s_i t} y_i(t)) = e^{-s_i t} P(t) g_i, \quad i = 1, 2, \dots, n \quad (92)$$

which by direct formal integration becomes

$$e^{-s_i t} y_i(t) = y_i(0) + \int_0^t e^{-s_i t'} P(t') g_i dt', \quad i = 1, 2, \dots, n \quad (93)$$

or

$$y_i(t) = e^{s_i t} y_i(0) + \int_0^t e^{s_i(t-t')} P(t') g_i dt' \quad i = 1, 2, \dots, n \quad (94)$$

In vector notation, Equation (94) is given by

$$\bar{y}(t) = e^{Dt} \bar{y}(0) + \int_0^t e^{D(t-t')} P(t') \bar{g} dt' \quad (95)$$

where

$$e^{Dt} = \begin{bmatrix} e^{s_1 t} & & & \\ & e^{s_2 t} & & \\ & & \ddots & \\ & & & e^{s_n t} \end{bmatrix}$$

Since $P(t)$ is a piecewise constant function changing only at discrete instances of time

$$P(t) = P(kT) = P_k \quad kT \leq t < (k+1)T \quad k = 0, 1, 2, \dots$$

we are mainly interested in the state of the system at discrete instances of time. The discrete analog of Equation (95) is

$$\bar{y}((k+1)T) = e^{DT} \bar{y}(kT) + P_k \int_0^T e^{D(t-t')} \bar{g} dt' \quad (96)$$

$$kT \leq t < (k+1)T$$

Now, by defining

$$G(T) = e^{DT}$$

and

$$\bar{h}(T) = \int_0^T e^{D(t-t')} \bar{g} dt' ,$$

equation (96) may be written as

$$\bar{y}((k+1)T) = G(T) \bar{y}(kT) + P_k \bar{h}(T), \quad kT \leq t < (k+1)T \quad (97)$$

Equation (97) is a vector difference equation defining the state of the system at discrete instances of time.

Assume the system is initially in the equilibrium state

$$\frac{d\bar{y}(0)}{dt} = \bar{y}(0) = \bar{0} \quad (98)$$

and let us follow the evolution of the system through state space when the driving function is a sequence of k sufficiently short pulses P_0 ,

P_1, P_2, \dots, P_{k-1} . At the completion of the first pulse, the state of the system is given by

$$\bar{y}(T) = P_0 \bar{h}(T) \quad , \quad (99)$$

which becomes the initial condition for the next pulse interval. Following the second pulse, the system is in the state

$$\bar{y}(2T) = G(T) P_0 \bar{h}(T) + P_1 \bar{h}(T) \quad , \quad (100)$$

which becomes the initial condition for the next pulse interval. At the completion of the third pulse, the state of the system becomes

$$\bar{y}(3T) = G(T)^2 P_0 \bar{h}(T) + G(T) P_1 \bar{h}(T) + P_2 \bar{h}(T) \quad . \quad (101)$$

Now, we may write an expression for the state of the system at the termination of the pulse sequence,

$$\begin{aligned} \bar{y}(kT) = & G(T)^{k-1} P_0 \bar{h}(T) + G(T)^{k-2} P_1 \bar{h}(T) + \dots \\ & + G(T) P_{k-2} \bar{h}(T) + P_{k-1} \bar{h}(T) \quad . \end{aligned} \quad (102)$$

The multiplication of a matrix $G^i(T)$ by a column vector $\bar{h}(T)$ yields a column vector. It can be seen from Equation (102) that the state of the system at the completion of the k^{th} pulse is given by the vector sum of k vectors. For a system with real distinct eigenvalues, the k vectors are linearly independent for $k \leq n$ because of the exponential nature of $G(T)$.²¹ If the system has complex eigenvalues, let them be $s_1 = -(\sigma + j\omega_r)$ and $s_j = -(\sigma - j\omega_r)$, the vectors will be linearly independent for $k \leq n$ if, and only if

$$e^{-(\sigma + j\omega_r)T} \neq e^{-(\sigma - j\omega_r)T} \quad , \quad (103)$$

which requires

$$\omega_r^T \neq \pi_m, \quad (104)$$

where m is an integer and ω_r is the imaginary part of any complex eigenvalue.²² Since we have restricted the pulses to be sufficiently short, the k vectors will be linearly independent for $k \leq n$.

If n vectors v_1, v_2, \dots, v_n are linearly independent, the only numbers m_1, m_2, \dots, m_n for which

$$m_1 v_1 + m_2 v_2 + \dots + m_n v_n = \bar{0} \quad (105)$$

are the numbers $m_1 = 0, m_2 = 0, \dots, m_n = 0$.²³ Hence, the only sequence of k pulses, where $k \leq n$, capable of leaving the system in its equilibrium state is the trivial sequence

$$P_0 = 0, P_1 = 0, \dots, P_{k-1} = 0. \quad (106)$$

If $k = n + 1$, there would be $n + 1$ vectors which, of course, cannot be linearly independent in an n -dimensional space. This assures us that a sequence of $n + 1$ pulses is capable of exciting the system and then returning it to its equilibrium state, and that the minimal sequence contains $n + 1$ pulses.

When $k = n + 1$, Equation (102) becomes

$$\begin{aligned} \bar{y}((n + 1)T) = & G^n(T) P_0 \bar{h}(T) + G^{n-1}(T) P_1 \bar{h}(T) + \dots \\ & + G(T) P_{n-1} \bar{h}(T) + P_n \bar{h}(T). \end{aligned} \quad (107)$$

Now, assume the pulse sequence has been properly adjusted so that the system is returned to its equilibrium state at the completion of the pulse sequence,

$$\bar{y}((n+1)T) = \bar{0} \quad .$$

We may factor out $P(0)$ and $\bar{h}(T)$, neither of which are zero, in Equation (107), obtaining

$$G^n(T) + \frac{P_1}{P_0} G^{n-1}(T) + \dots + G(T) \frac{P_{n-1}}{P_0} + \frac{P_n}{P_0} I = \bar{0} \quad . \quad (108)$$

It can be shown that if $Q(G(T)) = 0$ where Q is a polynomial, then $Q(z) = 0$ for every eigenvalue z of $G(T)$.²⁴ It follows that if we replace the matrix $G(T)$ with the complex variable z , Equation (108) becomes the characteristic equation for the $G(T)$ matrix

$$z^n + \frac{P_1}{P_0} z^{n-1} + \dots + z \frac{P_{n-1}}{P_0} + \frac{P_n}{P_0} = 0 \quad . \quad (109)$$

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation which substantiates our statement.

The values of z found by solving Equation (109) are the eigenvalues of the vector difference equation of the system (z -plane poles of the system). They may be related to the eigenvalues of the differential equation of the system (s -plane poles of the system) by the transformation

$$s = \frac{1}{T} \ln z \quad . \quad (110)$$

Although we have uniquely located the eigenvalues of the difference equation, we experience difficulty determining the eigenvalues of the differential equation from them. The difficulty arises because the logarithm is a multiple valued function.²⁵ Equation (110) is properly

written as

$$s = \frac{1}{T} \ln |z| + j \frac{1}{T} (\text{ARG } z - 2\pi m), \quad m = 0, 1, 2, \dots \quad (111)$$

By equating the real and imaginary parts we obtain,

$$\sigma = \frac{1}{T} \ln |z| \quad (112)$$

and

$$\omega_r = \frac{1}{T} (\text{ARG } z - 2\pi m) \quad m = 0, 1, 2, \dots \quad (113)$$

To avoid difficulty, we would like to use only the principal value of $\ln z$. This would be permissible if the pulse width, T , is chosen such that

$$T < \frac{\pi}{\omega_r}, \quad (114)$$

where ω_r is the imaginary part of any complex eigenvalue of the system. This is the same conclusion arrived at in Chapter II for a second order system.

When the minimal sequence of sufficiently short pulses has been used to obtain the prescribed response, the transfer function of the system may be formulated from the roots of Equations (109). The transfer function relating the excitation $E(s)$ to the system response $R(s)$ is given by

$$\frac{R}{E}(s) = \frac{H}{(s - \frac{1}{T} \ln z_1)(s - \frac{1}{T} \ln z_2) \dots (s - \frac{1}{T} \ln z_n)} \quad (115)$$

where z_i $i = 1, 2, \dots, n$ are the roots of Equation (109). Only the principal value of each logarithm is to be used.

In this chapter, it has been shown mathematically that a pulse sequence can be used to formulate a mathematical model of a physical system of the class specified in Chapter I. It is shown that there are certain restrictions the pulse sequence must satisfy, however. They are:

1. The sequence must be the minimal sequence capable of forcing the system to yield the prescribed response. Hence, the number of pulses should be equal to the order of the system plus one.
2. The pulse widths, T , should satisfy

$$T < \frac{\pi}{\omega_r} \quad (114)$$

where ω_r is the largest imaginary part of any complex eigenvalue of the system.

The relationship between the pulse amplitudes and the eigenvalues of the differential equation of the system has been shown. If the pulse sequence satisfies the above conditions, the eigenvalues may be uniquely determined. One very important question which has not been answered, however, is "How practical is the method?". This will be discussed in the next chapter.

CHAPTER IV

ANALOG COMPUTER STUDY

Reasons For Using Simulation

The mathematical analysis has shown that a pulse sequence can be used to formulate a mathematical model of a physical system. Next, we would like to apply the method to a number of different systems to determine, at least qualitatively, the time required to perform the measurement and the accuracy that might be expected. Because of the difficulty of obtaining a wide variety of physical systems and instrumenting them properly, it was decided to simulate them on an analog computer. The analog computer can be programmed to represent a physical system in a very short time and it accepts a voltage driving function and produces a voltage proportional to the system response which was necessary for the tests. Also, the accuracy of the computer is within one percent which made the simulated system well defined. If actual physical systems had been used, there would have been the problem of evaluating them by another method and, if the results did not agree with those obtained by the pulse method, it would be difficult to determine which procedure was in error. Thus, the analog computer is ideally suited for investigations of this nature.

The Pulse Generator

To evaluate the pulse method, some means of pulse generation was needed. The pulse generator should be capable of generating a sequence

of at least four pulses with adjustable amplitudes. Because the sole reason for constructing the generator was to evaluate the method, not to develop a test instrument, very rudimentary means were used.

The heart of the generator was a standard telephone stepping relay. The stepping relay was actuated by a sensitive sigma relay. The sensitivity of the sigma relay was such that it could be driven by either a standard sine wave generator or by an oscillator simulated on the analog computer. A 45-volt battery which had a center tap was used for the pulse voltages. The center tap of the battery was grounded so that both positive and negative voltages were available. Four potentiometers were used to set the heights of the four pulses. Two of the potentiometers were connected between the positive battery terminal and ground and two were connected from the negative terminal to ground. The sliders of the potentiometers were connected to four consecutive terminals on the stepping relay. The sliders were connected so that adjacent terminals had opposite polarity. Another potentiometer was connected between the movable contact on the stepping relay and ground. The slider of this potentiometer was connected to one output terminal of the pulse generator. The other output terminal was connected to chassis ground.

The completed generator was capable of generating an alternating sequence of four voltage pulses. The magnitude of each pulse could be varied from zero to twenty volts. The stepping relay limited the minimum pulse width to 50 milliseconds. Although the pulse generator was satisfactory for its designed purpose, there was certainly room for considerable improvement.

Results of the Analog Computer Simulation

A number of systems typical of those used in the construction of control systems were simulated on the analog computer. Very simple systems were studied first with the hope that the knowledge gained studying the simpler system structures would be of value when studying the more complicated configurations.

The first system simulated was the simplest possible,

$$\frac{dr}{dt} + r = e(t) \quad (116)$$

This could represent the velocity of a mass damper system with a force driving function or the response of a first order lag filter. The response to a single pulse forcing function is shown in Figure 9a. Figure 9b shows the response to a two pulse sequence with the second pulse not large enough to drive the system back to its equilibrium state. Figure 9c shows the response with the second pulse properly adjusted. Figure 9d shows the response when the amplitude of the second pulse exceeds the proper amount.

When the response shown in Figure 9c was obtained, the height of each pulse was measured with a three place digital voltmeter. The pulse heights were found to be $P_0 = 4.79$ volts and $P_1 = -2.90$ volts. The width of each pulse was 0.500 seconds.

The eigenvalue of the difference equation of the system is found from the pulse heights by solving

$$z + \frac{P_1}{P_0} = 0 \quad (117)$$

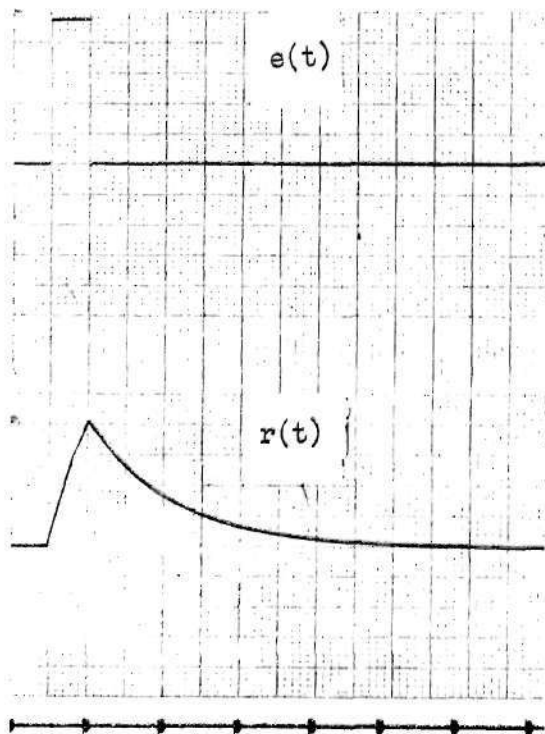


Figure 9a. Pulse Response of
A First Order System

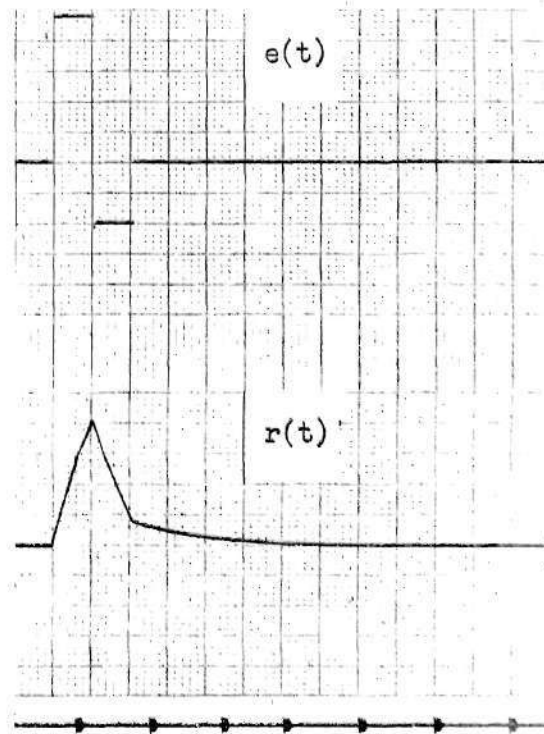


Figure 9b. Response When
Second Pulse is Too Small

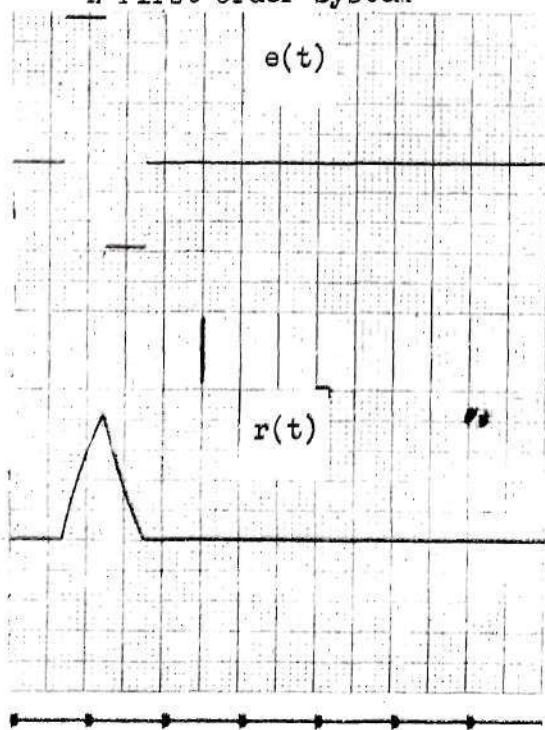


Figure 9c. Response When Second
Pulse is Properly Adjusted

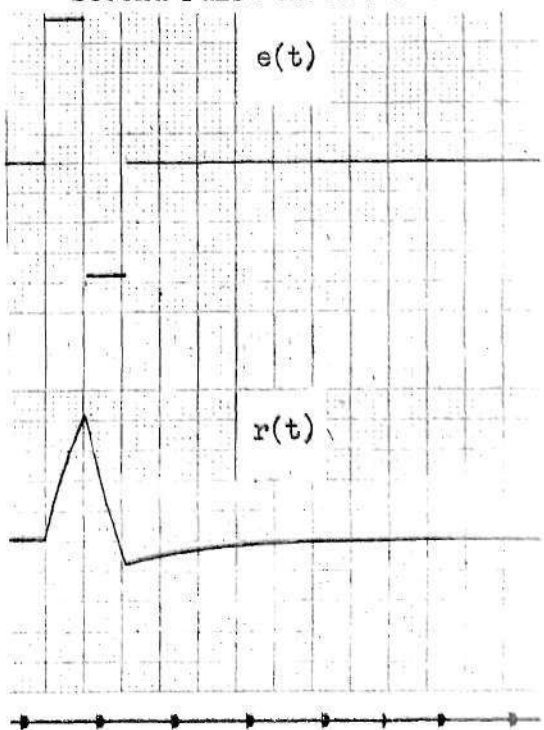


Figure 9d. Response When Second
Pulse is Too Large

When numerical values are substituted for the pulse heights, it is found that $z \approx 0.605$.

The transfer function of the system is formulated from the eigenvalues of the difference equation as shown by Equation (111). The transfer function relating the excitation to the system response of the first order system is

$$\frac{R}{E}(s) = \frac{H}{s - \frac{1}{T} \ln z_1} \quad (118)$$

$$\frac{R}{E}(s) = \frac{H}{s - 1/0.500 \ln (0.605)} \quad (119)$$

$$\frac{R}{E}(s) = \frac{H}{s + 1.004} \quad (120)$$

The differential equation may be formed from the transfer function. It is

$$\frac{dr}{dt} + 1.004 r = H e(t) \quad (121)$$

This example demonstrates how the mathematical model of the system is formulated from the pulse heights. The accuracy of the example is typical of the accuracy obtainable by this method.

In terms of the s-plane, increasing the height of the second pulse moves a signal zero along the negative real axis from minus infinity toward plus infinity. Actually, it moves an infinite column of zeros

from minus infinity toward plus infinity. However, only the zero on the real axis is important to the present discussion. As the second pulse height is gradually increased (in the negative direction) from zero, a signal zero moves in from minus infinity toward the system pole. As the signal zero approaches the system pole, the residue in the pole becomes smaller. Thus, the amplitude of the response at the completion of the pulse sequence becomes smaller. When the signal zero coincides with the system pole, the residue in the pole is made zero and the amplitude of the response at the completion of the second pulse is zero. If we continue to increase the second pulse height, the signal zero sweeps across the system pole and moves on toward the origin. When the zero has passed the system pole, the residue in the pole changes signs and becomes negative. Thus, the amplitude of the response at the completion of the second pulse will be negative. This is verified by the computer results. Actually, it can be shown²⁶ that for a linear system to yield a pulse response, it is necessary that the driving function contain zeros that cancel all the system poles.

The second system that was simulated is described by the second order differential equation,

$$\frac{d^2r}{dt^2} + \frac{dr}{dt} = e(t) \quad (122)$$

This is the mathematical model often used to describe servomotors. Figure 10a shows the response and its derivative to a short pulse. As can be seen in the figure, the pulse response does not tend to return to its original equilibrium state. This type of system has an infinite number

of equilibrium states. The large number of equilibrium states arises because the system has a zero eigenvalue. In normal co-ordinates, the system may be described by the two first order differential equations

$$\frac{dy_1(t)}{dt} = -y_1(t) + e(t) \quad (123)$$

and

$$\frac{dy_2(t)}{dt} = e(t) \quad (124)$$

Integration of the second equation yields

$$y_2(t) = \int_0^t e(t') dt' \quad (125)$$

If we think of this equation as describing a point moving along an eigenvector in state space, the point will move in proportion to the integral of the driving function. The tendency of electrical motors to integrate the forcing function is well known.

Figure 10b shows the response and the derivative of the response when the pulse sequence is adjusted so that the state space point governed by Equation (123) is left in its equilibrium state. From the pole-zero point of view, we would say that a signal zero cancels the system pole on the negative real axis.

Figure 11a shows the response and its derivative when the forcing function is adjusted to leave the state space point governed by Equation (124) in its original equilibrium state. Notice that following the completion of the pulse sequence, the response has the form

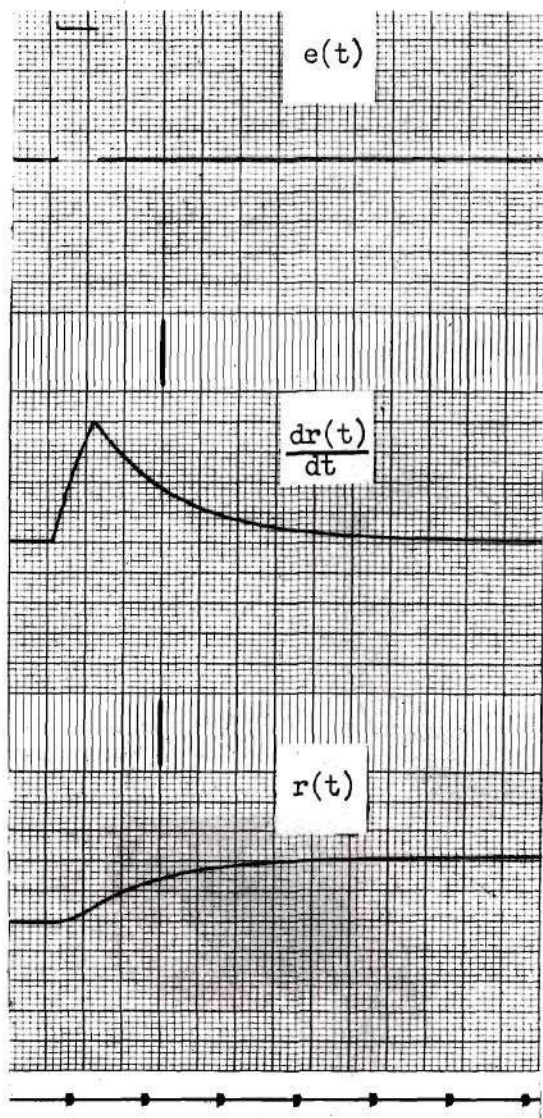


Figure 10a. Pulse Response of a System Having One Negative Real and One Zero Eigenvalue

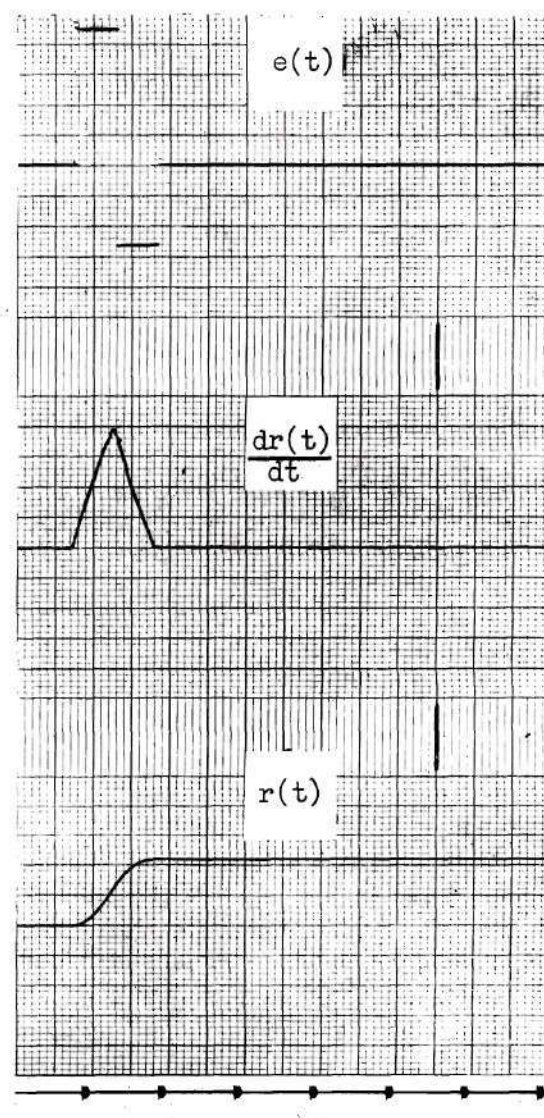


Figure 10b. Response When Pulse Sequence is Adjusted to Cancel Real Eigenvalue

$$ke^{s_1 t},$$

which is the free motion of the transient mode governed by Equation (123). It is certainly interesting to see that the forcing function can be adjusted to leave the individual modes unexcited and, thus, removed from the free behavior of the system following the pulse sequence. This gives us a very real verification of the validity of the seemingly abstract manipulations of the system in normal co-ordinate state space. Figure 11b shows the system response and its derivative when the pulse sequence is adjusted to leave the system in its original equilibrium state. Of course, an additional pulse was required to leave both modes unexcited.

The pulse amplitudes which obtained the response of Figure were measured to formulate a mathematical model of the system. The pulse amplitudes were $P_0 = 9.68$ volts, $P_1 = -15.5$ volts and $P_2 = 5.8$ volts. The width of each pulse was 0.5 seconds. To formulate the mathematical model, the eigenvalues of the difference equation are first found by solving

$$z^2 + \frac{P_1}{P_0} z + \frac{P_2}{P_0} = 0 \quad (126)$$

By substituting the measured values of pulse heights into Equation (126) we obtain

$$z^2 - 1.6 z + 0.6 = 0, \quad (127)$$

or

$$(z - 1.0)(z - 0.6) = 0. \quad (128)$$

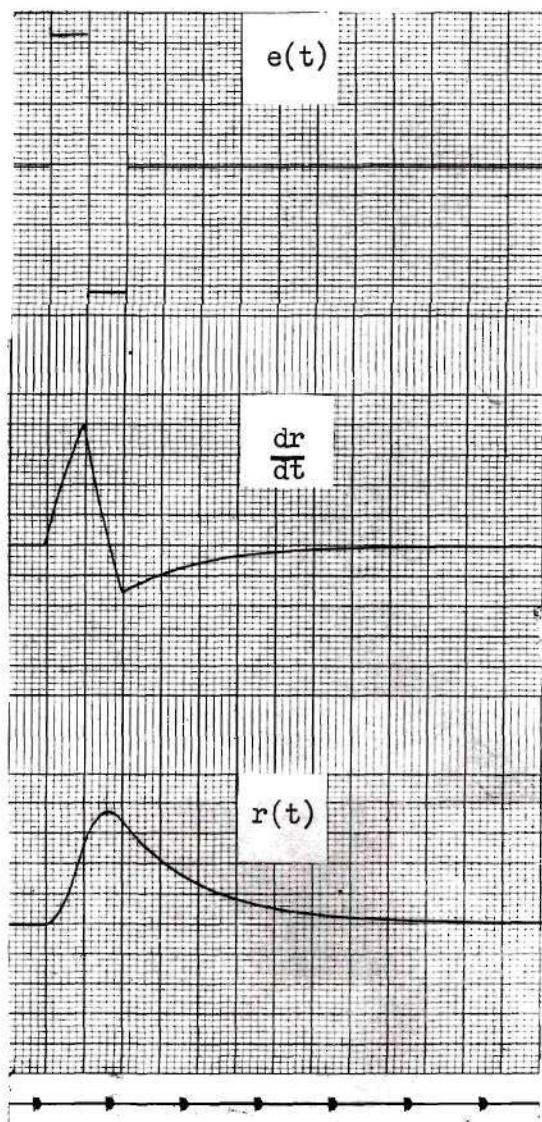


Figure 11a. Response When Pulse Sequence is Adjusted to Cancel Zero Eigenvalue

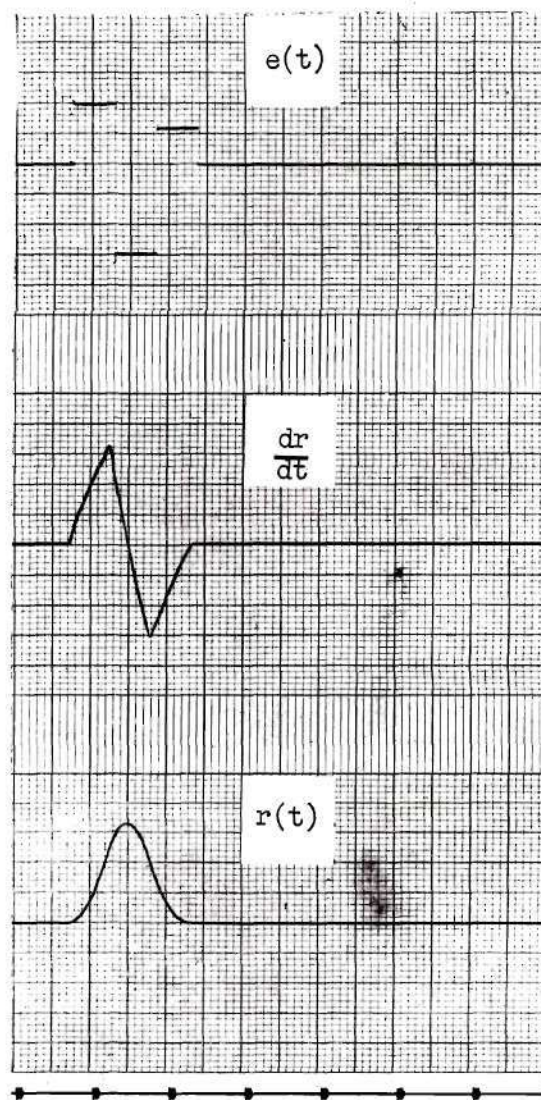


Figure 11b. Response When Pulse Sequence is Adjusted to Cancel Both Eigenvalues

The transfer function of the system is formulated from the characteristic values of Equation (127) using

$$\frac{R}{E}(s) = \frac{H}{\left(s - \frac{1}{T} \ln z_1\right) \left(s - \frac{1}{T} \ln z_2\right)} \quad (129)$$

A substitution of numerical values into Equation (129) yields

$$\frac{R}{E}(s) = \frac{H}{(s - 1/0.5 \ln 1.0)(s - 1/0.5 \ln 0.6)} \quad (130)$$

which reduces to

$$\frac{R}{E}(s) = \frac{H}{s(s + 1.01)} \quad (131)$$

The differential equation of the system may be formulated from the transfer function to give

$$\frac{d^2 r}{dt^2} + 1.01 \frac{dr}{dt} = H e(t) \quad (132)$$

The differences between the differential equation simulated on the analog computer and that formulated from the pulse heights are contributed to the following factors:

1. The accuracy of the analog computer was one percent.
2. The three place digital voltmeter used to measure the pulse heights introduced roundoff errors.
3. A slide rule was used to perform numerical calculations.

The results were very repeatable and the accuracy of the examples is

typical of that which can be obtained by this method.

The next system to be discussed is governed by the differential equation

$$\frac{d^2 r}{dt^2} + \frac{dr}{dt} + r = e(t) \quad (133)$$

This is an underdamped system having complex conjugate eigenvalues. Figure 12a shows the response to a short pulse and the corresponding derivative of the response. The oscillatory response is, of course, typical of complex eigenvalues. Figure 12b shows the response and its derivative when the pulse sequence is adjusted to leave the system unexcited.

The last system to be discussed is governed by the differential equation

$$\frac{d^3 r}{dt^3} + 0.4 \frac{d^2 r}{dt^2} + \frac{dr}{dt} = e(t) \quad (134)$$

This system has two complex conjugate eigenvalues denoting an oscillatory mode and a zero eigenvalue. An equation of this type quite often arises in the design of pneumatic control systems. Figure 13a shows the response and its two derivatives to a short pulse forcing function. As expected, the complex eigenvalues give rise to an oscillatory mode and the zero eigenvalue tends to integrate the driving function.

A system of this type could be represented in a 3-dimensional state space. The zero mode would be represented by a point moving along one co-ordinate in proportion to the integral of the excitation. The two

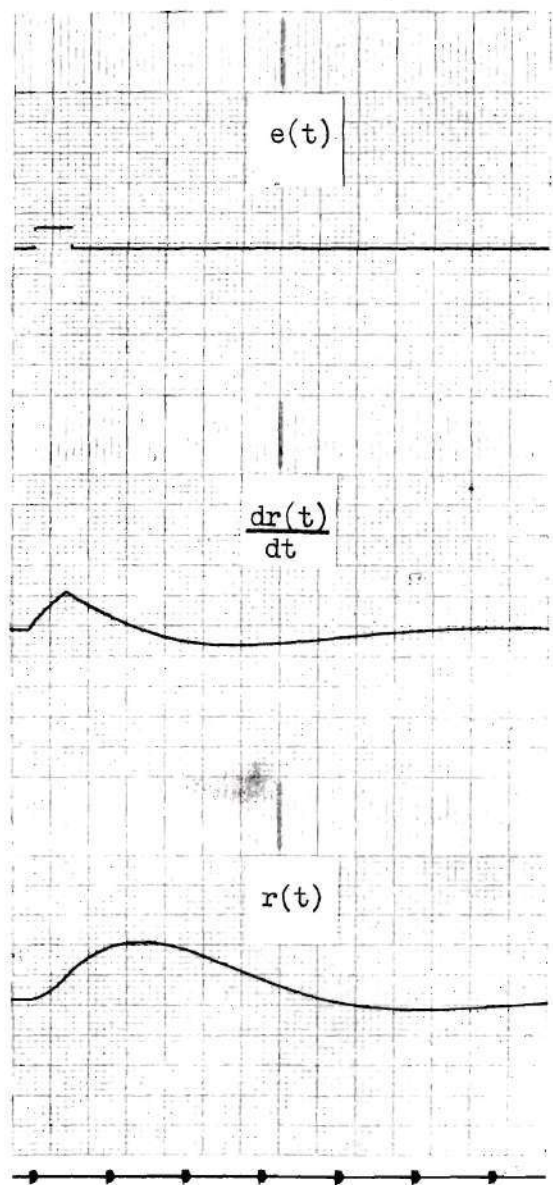


Figure 12a. The Response of a System Having Complex Eigenvalues To a Pulse Driving Function

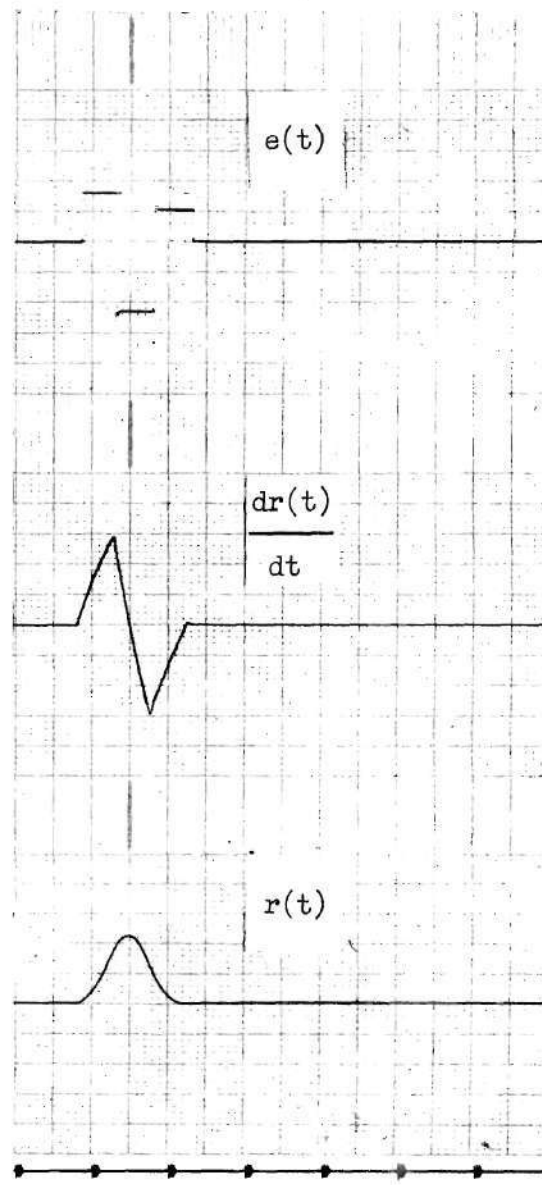


Figure 12b. The Response An Under Damped Second Order System to a Properly Adjusted Pulse Sequence

complex modes could be combined so that their motion was confined to the plane of the other two co-ordinates. Figure 13b shows the system response and its two derivatives when the forcing is adjusted to leave the oscillatory mode unexcited. It never ceases to be surprising that a system with a lightly damped mode of oscillation can be driven by a sequence of short discontinuous pulses which end abruptly in a sharp discontinuity and, yet, no oscillations appear in the response. From the Laplace-transform point of view, we would say that the forcing function contained zeros which cancelled the complex poles of the system.

Needless to say, adjustment of the pulses is very critical. If there exists a small maladjustment in the pulse sequence, the oscillatory mode will appear in the response. This, of course, is desirable. It allows us to accurately determine the value of the complex eigenvalues.

Procedure For Performing Pulse Measurement

The following test procedure seems to give the most accurate results in the least time. It was arrived at by evaluating a number of systems on the analog computer.

1. The pulse generator is connected to the system input terminals and the system response shown on an oscilloscope.
2. A single pulse is applied to the system to determine qualitatively the order of the system and the frequency of the oscillatory modes.
3. The number of pulses to be used is made equal to the order of the system plus one and the pulse widths should be made sufficiently small. When there is doubt as to the order of the system, as there will almost always be, $m + 1$ pulses should be used where m is the lowest possible

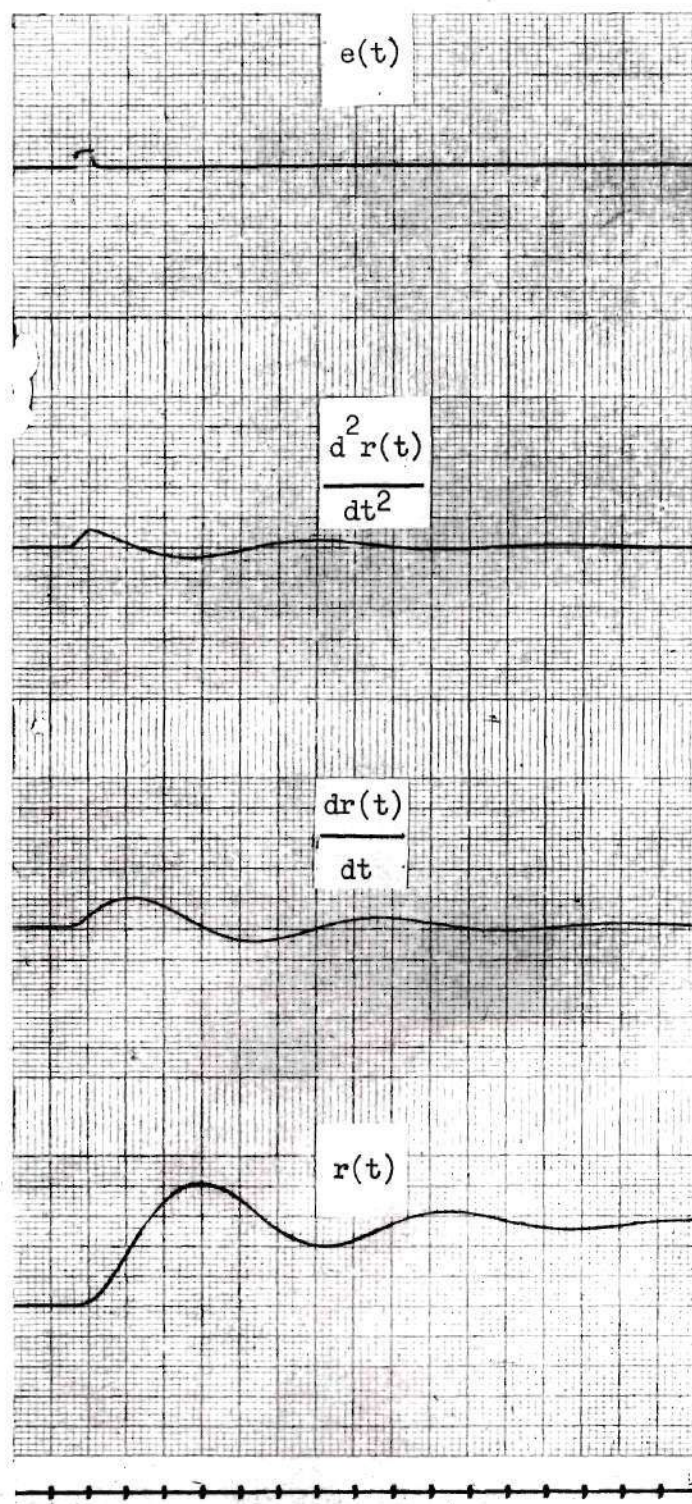


Figure 13a. The Response of a System
Having One Zero and Two Complex
Eigenvalues to a Pulse
Driving Function

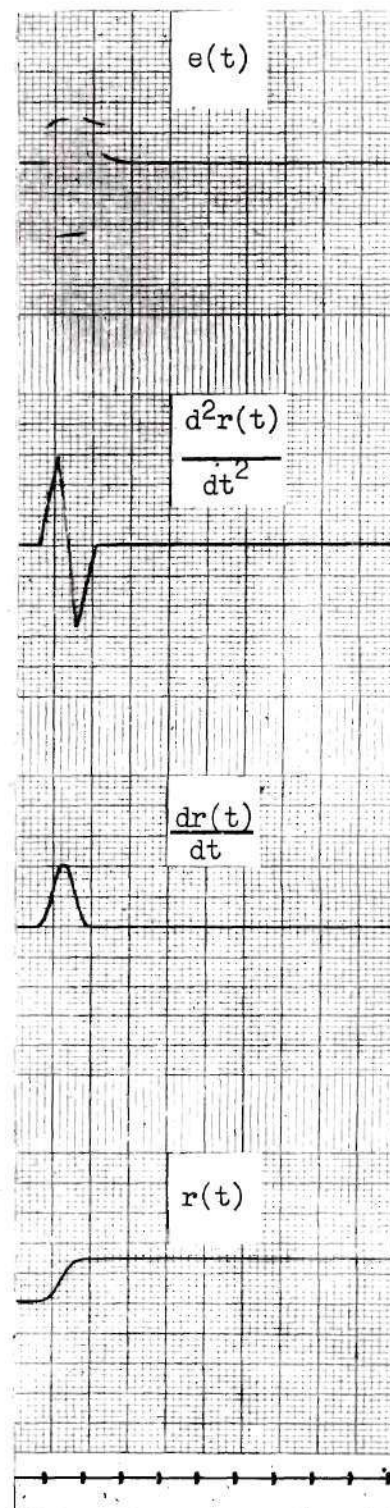


Figure 13b. The Response
to a Pulse Sequence
Adjusted to Leave the
Oscillatory Mode Unexcited

order of the system. If too few pulses are used, no harm is done for it will be impossible to return the system to its equilibrium state. This will soon become apparent and an additional pulse can be added to the sequence. However, if the sequence contains too many pulses, it will be possible to return the system to equilibrium and an erroneous answer will be obtained. The result will not be as erroneous as might at first be expected, however, for the sequence must still leave each of the system's modes unexcited. The results will contain all the system's modes plus one additional mode for each pulse in excess of the minimal number. It is not difficult to determine a satisfactory pulse width. If the pulses are too long, the system response will have time to oscillate during a pulse interval. If the pulses are made short enough so that the response does not oscillate during a pulse interval, there is little chance of difficulty.

4. The system is excited by the pulse sequence and the response observed. The pulse heights are then adjusted, the system allowed to return to equilibrium, and the sequence reapplied. Clues as to how the sequence should be adjusted may be obtained by observing the response. For second and higher order systems, the response will resemble a Gaussian probability function when the pulse sequence is properly adjusted. Thus, the response should reach a maximum approximately in the center of the pulse sequence and it should be reasonably symmetrical about the maximum point. Knowing this, it is not too difficult to tell what adjustments are needed.

5. Once the pulse sequence has been adjusted, the sensitivity of the oscilloscope should be increased. Although it will no longer be

possible to observe the total response because it will be off the screen, it will be possible to closely observe the motion that takes place after the completion of the pulse sequence. In this way, fine adjustments may be made to the pulse heights that will improve the accuracy of the measurement.

6. Once the pulse sequence has been found which produces the prescribed response, the generator may be disconnected from the system and the height of each pulse measured with a voltmeter. This is done by stopping the sequence at each individual pulse.

7. Next, the characteristic equation for the discrete transition matrix is written using the measured values of the pulse heights. For a sequence of $n + 1$ pulses, the equation is

$$P_0 z^n + P_1 z^{n-1} + \dots + P_{n-1} z + P_n = 0 \quad (135)$$

Equation (135) is factored yielding the eigenvalues of the difference equation

$$z = z_1, z_2, \dots, z_n$$

8. The transfer function of the system is formulated from the roots of Equation (135) and the pulse widths as shown

$$\frac{R}{E}(s) = \frac{H}{(s - \frac{1}{T} \ln z_1)(s - \frac{1}{T} \ln z_2) \dots (s - \frac{1}{T} \ln z_n)} \quad (136)$$

where it is understood that only the principal value of the logarithm is used. This completes the identification, at least within a constant multiplier.

Identification of a Servomotor

A large percentage of the control systems in use today are a-c or carrier systems. Information is transmitted through the system by means of a suppressed carrier amplitude modulated signal. A two phase servomotor performs the demodulation with the motor shaft moving in proportion to the modulating signal. Since so many systems are of this type, it would be highly desirable to be able to identify their dynamic characteristics also.

It would be possible to apply the pulses to a modulator and obtain a-c pulses at the modulator output. However, it would then be necessary to include the dynamic characteristics of the modulator in the analysis. The carrier pulses could be obtained by replacing the battery of the pulse generator with an a-c reference signal.

To evaluate the possibility of identifying a-c systems, a carrier system was set up consisting of an a-c amplifier driving a two phase a-c servomotor. A potentiometer excited by a center tapped battery was connected to the motor shaft. The voltage between the potentiometer slider and the center tap of the battery was proportional to the motor shaft position and was shown on an oscilloscope. The pulse generator was connected to the input of the amplifier. It was found that three pulses were all that was necessary to excite the motor and then return it to its equilibrium state. The mathematical model formulated from the pulse amplitudes agreed quite well with data obtained from a pulse response test.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

A pulse method of formulating a mathematical model of a linear system was proposed in Chapter I. It was shown in Chapter III that if the pulse sequence satisfied certain conditions, it was possible to uniquely determine the eigenvalues of the system from the pulse amplitudes. The conditions the pulse sequence should satisfy were:

1. The pulse sequence should excite the system and then return it to its equilibrium state coincident with the completion of the sequence.
2. The pulse sequence should be the minimum sequence capable of producing the prescribed response. It was shown that the minimal sequence would contain $n + 1$ pulses, where n is the order of the system.
3. The pulse width, T , should satisfy

$$T < \frac{1}{2f_r} \quad (137)$$

where f_r is the highest frequency appearing in the system's impulse response.

The analog computer study was discussed in Chapter IV. The computer study verified the mathematical results and also showed the method to be of practical value. The pulse method is capable of very good accuracy, probably better than either the transient or the frequency response methods. The time required for system identification by the pulse method appears to be between that required by transient tests and

that necessary for steady-state frequency response tests (for the class of systems studied).

The major disadvantages of the pulse method are the difficulties which arise when the system contains zeros, and the excessive time which would probably be required to identify systems of fourth order and higher. There exists a possibility of overcoming each of the difficulties. An adjustable filter could be added to the pulse generator. By passing the pulse sequence through the filter, it would be possible to generate a signal whose Laplace-transform contained both poles and zeros. The practical aspects of such a procedure are not known. It is possible that after sufficient experience, the pulse method could be applied to higher order systems in a reasonable amount of time.

The concepts discussed in this thesis have some rather interesting implications in other fields. Diamond and Gerst²⁷ have recently discussed the possibility of shaping the input pulse to a communication channel to eliminate the asymptote which generally exists at the channel output following a short pulse. Although their problem was considerably different than the problem considered in this thesis, and they used a different mathematical approach, some of their conclusions are identical with those here. For instance, they show that a pulse of width a may be obtained from a linear communication channel by driving the input with a sequence of $n + 1$ (n is the order of the channel) properly adjusted pulses of width, T . They show that since $a = (n + 1)T$, it is possible to make a arbitrarily small. This same conclusion was reached during the work of this thesis prior to the publication of their paper.

Another interesting possible application for the tuned pulse sequence concept is in the field of sampled-data control systems.

A typical sampled-data control system periodically samples its error signal* and, by means of a hold circuit, applies the error signal to the process being controlled until the next error sample is taken. Applying the sampled error to the process forces the process to respond in the proper direction to reduce the error. The deadbeat response of the simulated motor shown in Figure 10b suggests the possibility of applying a sequence of pulses proportional to the sampled error signal to the process, rather than the error itself. The pulse sequence could be tuned to cancel the undesirable load dynamics and, thus, force the process to respond quickly without excessive lags or overshoot. If desired, the response shown in Figure 10b could be obtained in less time by making the pulse widths shorter and the amplitudes larger. It almost seems that if one continued to make the pulses shorter and higher, eventually the motor could be made to respond instantaneously. Unfortunately, the pulse amplitudes would become large enough to invalidate the linearity assumption (the motor would saturate) and a further increase in the pulse height would not result in an additional decrease in response time. However, it should be possible to obtain a significant improvement in system performance by this means. Another difficulty which occurs in sampled-data systems that might be eliminated by using a tuned pulse sequence is the problem of high frequency oscillations between sampling periods. As shown in Figure 13b, it is possible to obtain deadbeat response from a lightly damped system by means of a properly adjusted

*The error signal is defined as the difference between the command input and the system output.

pulse sequence forcing function.

There are a number of interesting possibilities of the concept of using pulse sequences to formulate a mathematical model of a physical system which would bear further study. An electronic pulse generator could be built capable of generating considerably shorter pulses than the generator used in this thesis. Then, it would be possible to identify much faster responding systems than those studied here. This would make it possible to establish a practical upper limit on the system bandwidth beyond which this method would not be suitable.

There are some interesting theoretical questions involved when considering the possibility of using a-c pulses to evaluate a-c systems. Although impulse and step functions have been used for years to identify a-c servomechanisms, signals of this nature contain an infinite frequency spectrum. This results in an overlapping in the spectrum of the modulated signal which would prevent the demodulated output variable from uniquely representing the input. Experimental investigations of a-c systems do not reveal any difficulties, so perhaps the question is of little practical importance. It is felt, however, that additional work is necessary before the pulse method could be used with confidence to identify a-c systems.

It should be possible to develop a method of system identification using a sequence of constant amplitude pulses with adjustable widths. Such a method would have some advantages over the method proposed here in that it could be of aid in the identification of some classes of nonlinear systems. Many nonlinear systems may be represented in block diagram form by a nonlinear block followed by a linear block. If the

nonlinearity is symmetrical, as is usually the case, a constant amplitude pulse sequence might be used to identify the linear portion of the system. Another class of nonlinear systems is characterized by several piecewise linear regions. By varying the amplitude of the constant amplitude pulse sequence, it should be possible to formulate several linear models, each valid for a certain region of input amplitudes.

Although the method proposed here certainly is not a complete solution to the linear system identification problem, it could be developed into a valuable method of linear system identification.

APPENDIX A

The purpose of this appendix is to outline, by means of an example, the procedure for determining the matrix T which transforms a system into normal co-ordinates. The example will show the geometric interpretation of normal co-ordinates and how it is possible for a forcing function to leave individual modes unexcited.

Example of a third order system in state space:

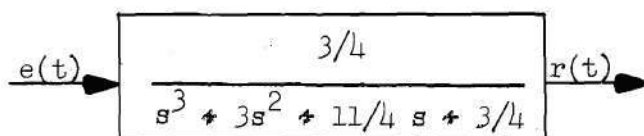


Figure A1. Linear Third Order System

The system is governed by the differential equation

$$\frac{d^3 r}{dt^3} + 3 \frac{d^2 r}{dt^2} + \frac{11}{4} \frac{dr}{dt} + \frac{3}{4} r = \frac{3}{4} e(t) \quad (A1)$$

To study this system in state space, let

$$x_1 = r$$

and

$$\frac{dx_1}{dt} = x_2 \quad (A2)$$

$$\frac{dx_2}{dt} = x_3$$

$$\frac{dx_3}{dt} = -3/4 x_1 - 11/4 x_2 - 3 x_3 + 3/4 e(t)$$

Vector notation yields

$$\frac{d\bar{x}(t)}{dt} = A \bar{x}(t) + P(t) \bar{b} \quad (A3)$$

where

$$\bar{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3/4 & -11/4 & -3 \end{bmatrix} \quad \text{and } P(t) \bar{b} = e(t) \begin{bmatrix} 0 \\ 0 \\ 3/4 \end{bmatrix}.$$

First consider the homogenous equation

$$\frac{d\bar{x}(t)}{dt} = A \bar{x}(t) \quad (A4)$$

Assume a solution of the form

$$\bar{x}(t) = e^{\lambda t} \bar{x}^{(1)} \quad (A5)$$

Substitution of Equation (A5) reduces Equation (A4) to

$$\lambda e^{\lambda t} \bar{x}^{(1)} = e^{\lambda t} A \bar{x}^{(1)}, \quad (A6)$$

or

$$\lambda \bar{x}^{(1)} = A \bar{x}^{(1)}, \quad (A7)$$

and by rearrangement

$$(\lambda I - A) \bar{x}^{(1)} = \bar{0}, \quad (A8)$$

where I is the unit matrix. For $\bar{x}^{(1)}$ to exist, it is necessary that

$$|\lambda I - A| = 0, \quad (A9)$$

which leads to the characteristic equation

$$\lambda^3 + 3\lambda^2 + 11/4\lambda + 3/4 = 0. \quad (A10)$$

By solving,

$$\lambda = -1/2, -1, -3/2.$$

These are the eigenvalues of the system. Once the eigenvalues are known, their eigenvectors may be found by solving the equation

$$[\lambda_i I - A] \bar{x}^{(i)} = 0, \quad \lambda_i = -1/2, -1, -3/2. \quad (A11)$$

For $\lambda = -1/2$, Equation (A11) becomes

$$\begin{bmatrix} -1/2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 3/4 & 11/4 & 5/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \quad (A12)$$

The three equations determine the eigenvector within a constant. Let

$x_1 = k_1$, then

$$-1/2 k_1 - x_2 = 0 \quad (A13)$$

$$-1/2 x_2 - x_3 = 0$$

$$3/4 k_1 + 11/4 x_2 + 5/2 x_3 = 0.$$

By solving,

$$\begin{aligned} x_2^{(1)} &= -1/2 k_1 \\ x_3^{(1)} &= -1/4 k_1 \end{aligned} \quad .$$

Thus,

$$\bar{x}^{(1)} = \begin{bmatrix} k_1 \\ -1/2 k_1 \\ 1/4 k_1 \end{bmatrix} \quad (A14)$$

In a similar manner, the eigenvectors for the other two eigenvalues may be found as

$$\bar{x}^{(2)} = \begin{bmatrix} k_2 \\ -k_2 \\ k_2 \end{bmatrix} \quad \text{and} \quad \bar{x}^{(3)} = \begin{bmatrix} k_3 \\ -3/2 k_3 \\ 9/4 k_3 \end{bmatrix} \quad (A15)$$

If the three eigenvectors are made the columns of a necessarily nonsingular matrix T

$$T = \begin{bmatrix} k_1 & k_2 & k_3 \\ -1/2 k_1 & -k_2 & -3/2 k_3 \\ 1/4 k_1 & k_2 & 9/4 k_3 \end{bmatrix}, \quad (A16)$$

then

$$T^{-1} AT = D \quad (A17)$$

where

$$D = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3/2 \end{bmatrix} \quad (A18)$$

To transform Equation (A3) into normal co-ordinates, let

$$\bar{x}(t) = T \bar{y}(t) \quad (A19)$$

Equation (A3) becomes

$$\frac{d\bar{y}(t)}{dt} = D \bar{y}(t) + P(t) \bar{g} \quad (A20)$$

where D is the 3×3 diagonal matrix defined by Equation (A18) and

$$\bar{g} = T^{-1} \bar{b} = \begin{bmatrix} 3/2 k_1 \\ -3/k_2 \\ 3/2 k_3 \end{bmatrix}$$

To make the normal co-ordinate Equation (A20) as simple as possible, choose k_1 , k_2 and k_3 to make

$$\bar{g} = \begin{bmatrix} 3/2 k_1 \\ -3/k_2 \\ 3/2 k_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 3/2 \end{bmatrix} \quad (A21)$$

Solving Equation (A21) gives $k_1 = 3$, $k_2 = -3$ and $k_3 = 1$. The transformation matrix becomes

$$T = \begin{bmatrix} 3 & -3 & 1 \\ -3/2 & 3 & -3/2 \\ 3/4 & -3 & 9/4 \end{bmatrix} \quad (A22)$$

Equation (A20) in explicit notation is given by

$$\frac{dy_i(t)}{dt} = \lambda_i y_i(t) + P(t)(-\lambda_i) \quad i = 1, 2, 3 \quad (A23)$$

The discrete solution is given by

$$y_i((k+1)T) = e^{\lambda_i T} y_i(kT) + P(kT)(1 - e^{\lambda_i T}) \quad (A24)$$

$$kT \leq t < (k+1)T \quad i = 1, 2, 3$$

The linear nonsingular transformation into normal co-ordinates has "uncoupled the transient modes". By means of Equations (A24), we may study the effect of the forcing function on each mode individually. Each of the three equations may be represented in state space by a point moving along the corresponding eigenvector. The state of the system at any time is given by the vector sum of the three vectors defined by the three points. Figure A1 shows the 3-dimensional state space. The unit vectors $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ form the orthogonal co-ordinate systems to which $\bar{x}(t)$ is referred. The unit vectors $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ are the eigenvectors of the A matrix. They form an oblique co-ordinate system to which $\bar{y}(t)$ is referred. A distinction must be made between the scale by which length is measured and the scales which are attached to the oblique co-ordinates²⁸. The scale of length is attached to the orthogonal co-ordinate system defined by the unit vectors $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. Each of

the oblique axes have their own scale. The scale does not measure length, but is used to determine the value of the projection of a point upon that axis.

Each of the three Equations (A22) govern the behavior of a point which moves along the corresponding eigenvector. Figure A2 shows the movement of the individual points with a sequence of two pulses driving the system. The sequence is adjusted to leave the second mode unexcited. It is easy to see that the unexcited mode will not appear in the free response of the system following the pulse sequence.

The behavior of the three modes along their respective eigenvectors is shown in Figure A3 with the pulse sequence adjusted to leave all three modes unexcited.

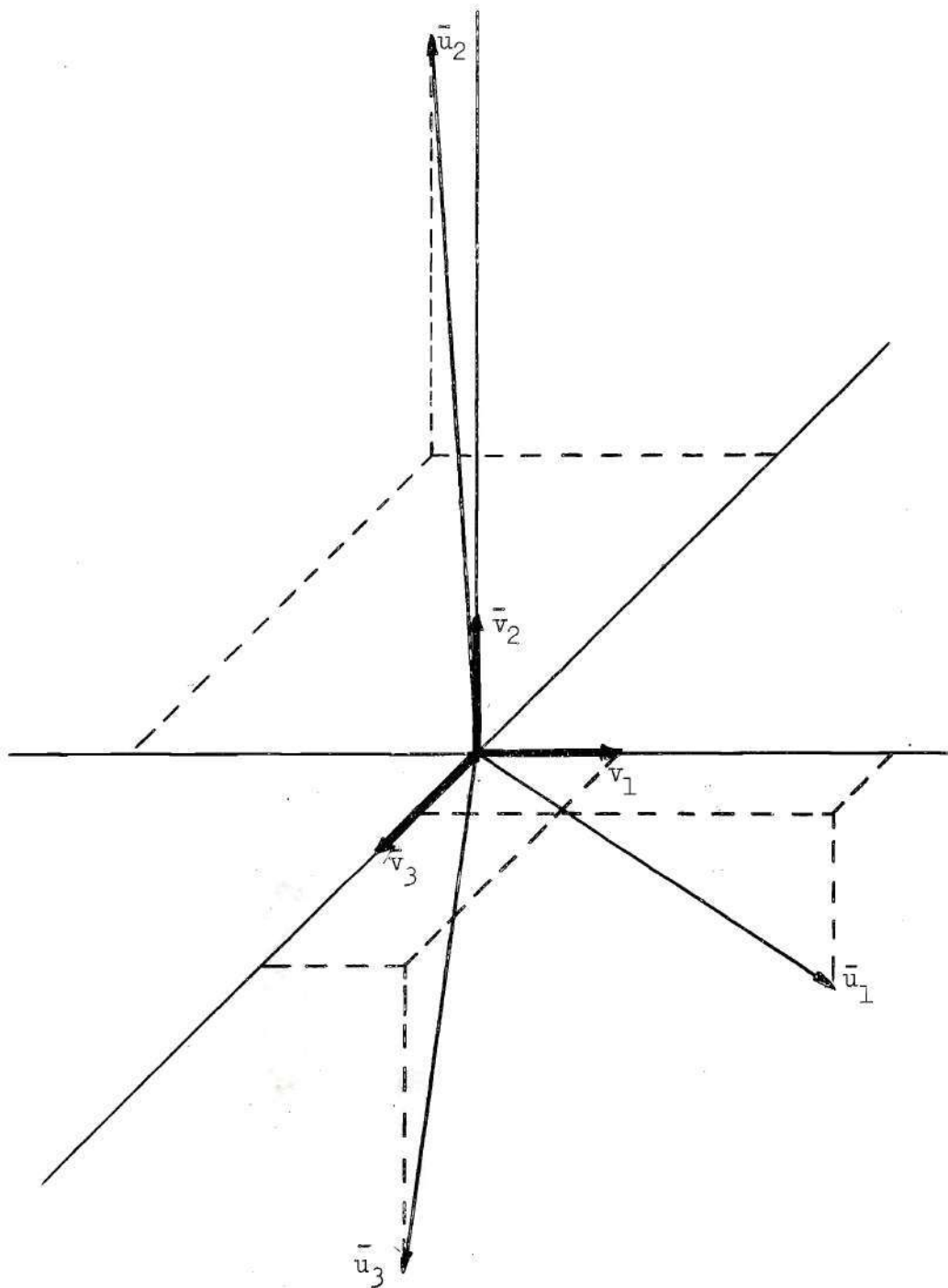


Figure A1. State Space Representation of
A Third Order System

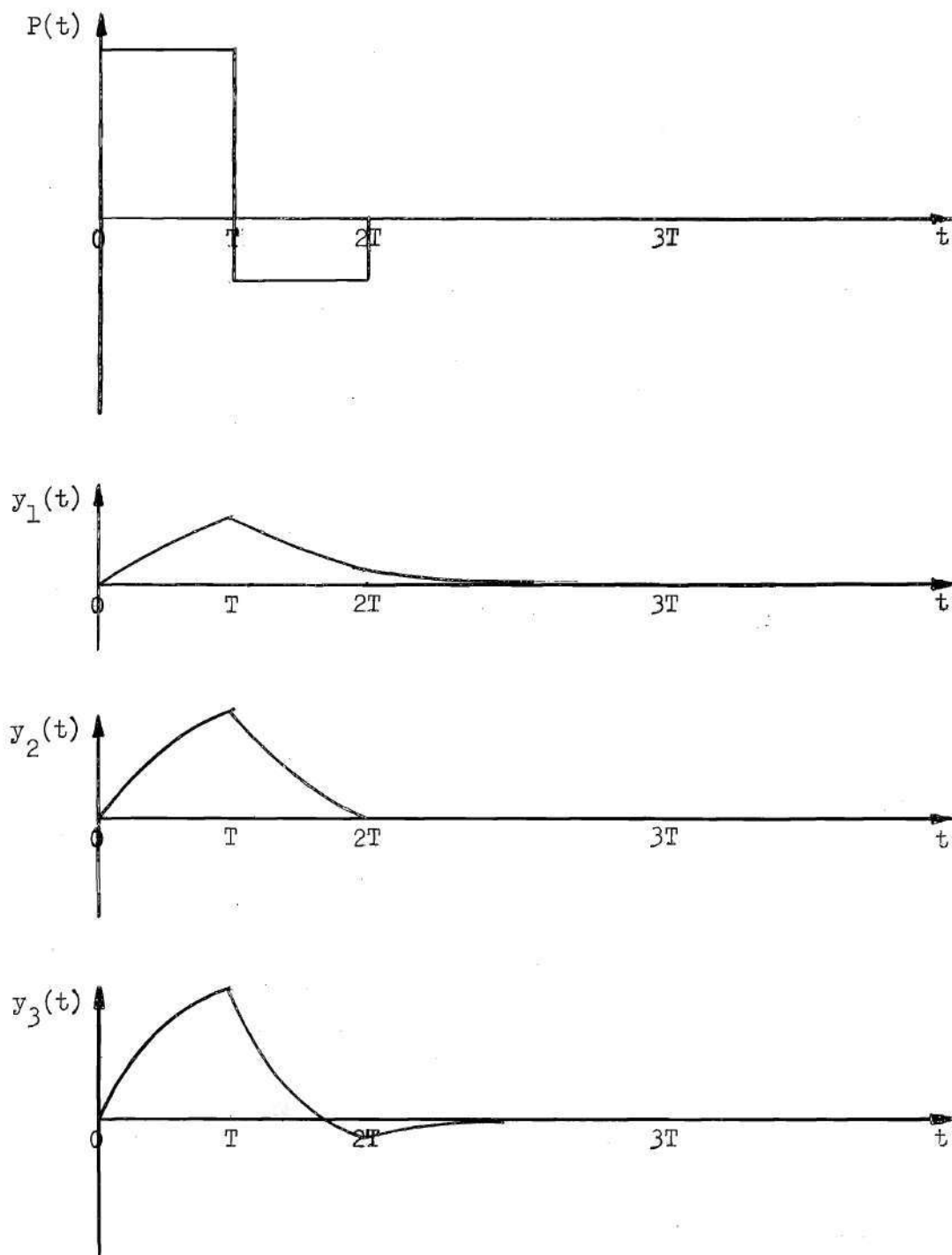


Figure A2. Behavior of Individual Modes With Pulse Sequence Adjusted To Leave $y_2(t)$ Unexcited

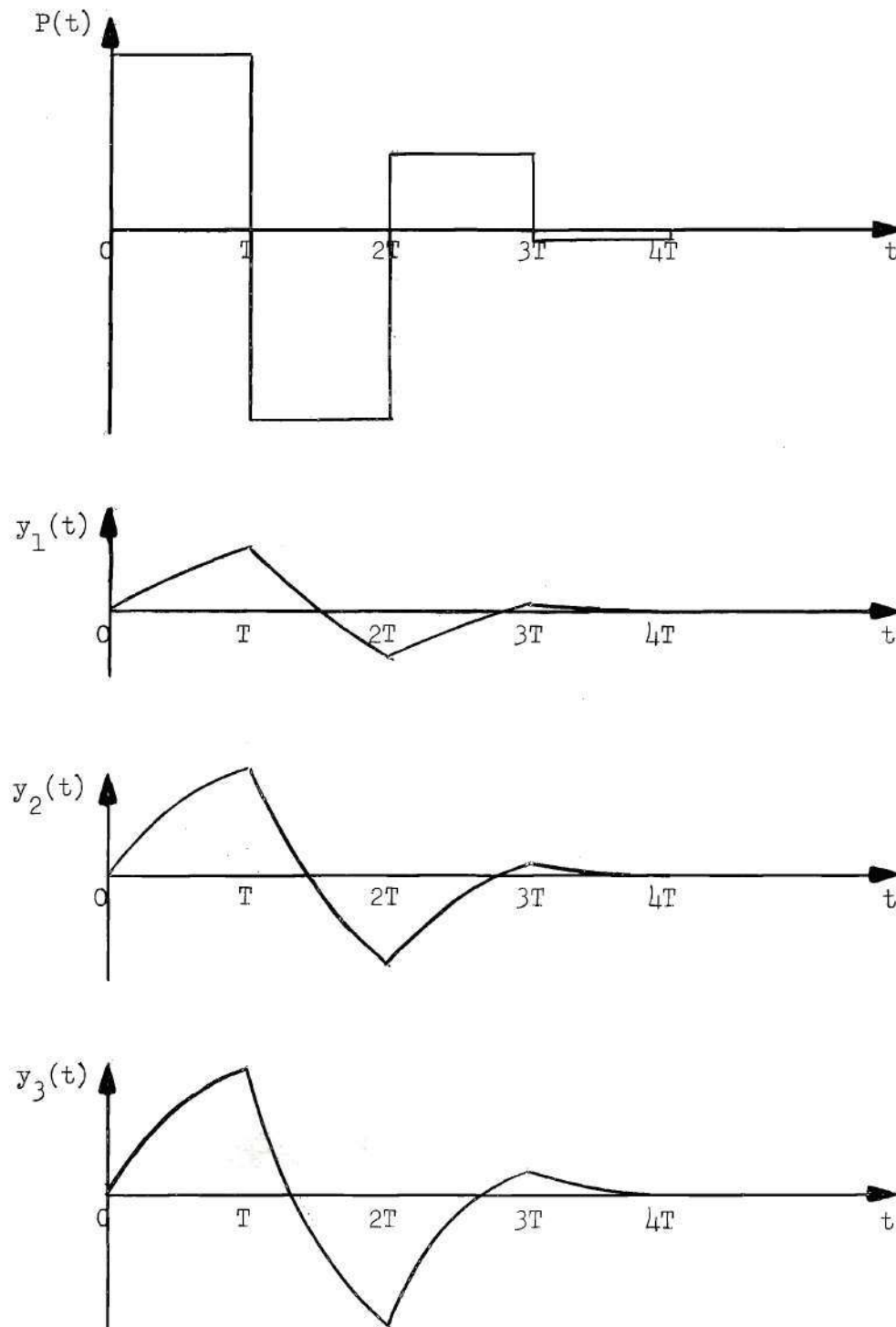


Figure A3. Movement of the Individual Modes Along Their
Respective Eigenvectors With A Properly Adjusted
Pulse Sequence Forcing Function

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